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ON ROLLE'S FUNCTION θ IN THE MEAN VALUE THEOREM FOR THE CASE OF THE NOWHERE DIFFERENTIABLE $f'(x)$ DEFINED BY DINI

BY

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In a recent paper* Prof. Ganesh Prasad has investigated the essential functional properties of $\theta(h)$ in the Mean Value Theorem for the case of nowhere differentiable $f'(x)$ and in particular he has considered the special case in which $f'(x)$ is the Weierstrass's non-differentiable function.

The object of the present paper is to study the nature of $\theta(h)$ when $f'(x)$ is the Dini's non-differentiable function $\sum\limits_1^\infty \frac{\sin(16^n x)}{2^n}$.

In § 1, I have obtained different values of θ for a particular constant value of h, *viz.*, $\frac{\pi}{16}$ and have shown that these values form an enumerable set.

In § 2, I have discussed the question: How does $\theta_1(h)$ behave as h varies?

In § 3, I have considered the question whether $\frac{d\theta}{dh}$ exists for $\theta = \frac{p}{16}$ ($p = 1, 2, \ldots\ldots 15, 16$) when h is of the form $\frac{2m\pi}{16}$ and also the value of $\frac{d\theta}{dh}$ at $h = 0$; here m is integral.

I take this opportunity to express my best thanks to Prof. Ganesh Prasad for his kind interest and constant encouragement.

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* *Bulletin of the Calcutta Mathematical Society*, Vol. XXIII, pp. 57-66.

§1.

Case of $h=\frac{\pi}{16}$.

1. In this case

$$f'(h)=\sum_{1}^{\infty}\frac{\sin(16^n h)}{2^n}.$$

$$\therefore\ f(h)=-\sum_{1}^{\infty}\frac{\cos(16^n h)}{2^n.16^n}.$$

In this article I will discuss the interesting application of Rolle's theorem to such a function. Thus putting $h=\frac{\pi}{16}$ and applying the Rolle's theorem, we obtain for θ, the equation,

$$\sum_{0}^{\infty}\frac{\sin(16^n\theta\pi)}{2^n}-\frac{2}{\pi}=0\equiv \mathrm{W}(\theta)\text{ say.}$$

Now we have to determine the values of θ for $h=\frac{\pi}{16}$ from the above equation and to ascertain whether in a finite interval (0, 1) there is any limiting point of the zeroes of $\mathrm{W}(\theta)$.

2. Applying the inequality $1>\frac{\sin\theta}{\theta}>\frac{2}{\pi}$, we find that at least one root θ lies in each of the intervals $(\frac{3}{16}, \frac{6}{16})$ and $(\frac{10}{16}, \frac{13}{16})$, or more approximately in the intervals $(\frac{3}{16}, \frac{4}{16})$ and $(\frac{12}{16}, \frac{13}{16})$.

Proof :—

$$\frac{2}{\pi}+\mathrm{W}(\tfrac{3}{16})=\sin\frac{3\pi}{16}<\frac{3\pi}{16}.\quad \text{But }\frac{3\pi}{16}<\frac{2}{\pi},$$

$$\therefore\ \mathrm{W}(\tfrac{3}{16})\text{ is negative.}$$

Also $\frac{2}{\pi}+\mathrm{W}(\frac{6}{16})=\sin\frac{6\pi}{16}>\frac{3}{4}.$ But $\frac{3}{4}>\frac{2}{\pi}$,

$$\therefore\ \mathrm{W}(\tfrac{6}{16})\text{ is positive.}$$

Hence there lies a root between $\theta=\frac{3}{16}$ and $\theta=\frac{6}{16}$. Similarly for the interval $(\frac{10}{16}, \frac{13}{16})$.

3. Further examining closely we find that roots in the interval $(\frac{3}{16}, \frac{4}{16})$ can be further separated by the following intervals,

$$\left\{ \frac{3+\frac{1}{2}+\frac{1}{2^2}+\cdots\cdots+\frac{1}{2^7}}{16}, \frac{3+\frac{1}{2}+\frac{1}{2^2}+\cdots\cdots+\frac{1}{2^8}}{16} \right\},$$

$$\left\{ \frac{3+\frac{1}{2}+\frac{1}{2^2}+\cdots\cdots+\frac{1}{2^8}}{16}, \frac{3+\frac{1}{2}+\frac{1}{2^2}+\cdots\cdots+\frac{1}{2^9}}{16} \right\},$$

$$\left\{ \frac{3+\frac{1}{2}+\frac{1}{2^2}+\cdots\cdots+\frac{1}{2^{10}}}{16}, \frac{3+\frac{1}{2}+\frac{1}{2^2}+\cdots\cdots+\frac{1}{2^{11}}}{16} \right\},$$

$$\left\{ \frac{3+\frac{1}{2}+\frac{1}{2^2}+\cdots\cdots+\frac{1}{2^{12}}}{16}, \frac{3+\frac{1}{2}+\frac{1}{2^2}+\cdots\cdots+\frac{1}{2^{13}}}{16} \right\},$$

$$\left\{ \frac{3+\frac{1}{2}+\frac{1}{2^2}+\cdots\cdots+\frac{1}{2^{13}}}{16}. \frac{3+\frac{1}{2}+\frac{1}{2^2}+\cdots\cdots+\frac{1}{2^{14}}}{16} \right\},$$

and the roots in the interval $(\frac{12}{16}, \frac{13}{16})$, by the interval,

$$\left\{ \frac{12+\frac{1}{2}+\frac{1}{2^2}+\cdots\cdots+\frac{1}{2^4}}{16}, \frac{12+\frac{1}{2}+\frac{1}{2^2}+\cdots\cdots+\frac{1}{2^5}}{16} \right\}.$$

Considering the intervals $(0, \frac{1}{16})$, $(\frac{1}{16}, \frac{2}{16})$,......$(\frac{15}{16}, 1)$, I have found roots, to the approximation of $\frac{1}{16}$, to lie in the intervals marked as $\frown\!\frown$ in the following and the signs of $W(\theta)$ are represented at the top of each interval.

$$\left(\overset{-}{0}, \overset{-}{\frac{1}{16}}\right), \cdots\cdots \overset{\frown\!\frown}{\left(\overset{-}{\frac{3}{16}}, \overset{+}{\frac{4}{16}}\right)}, \cdots\cdots \overset{\frown\!\frown}{\left(\overset{+}{\frac{12}{16}}, \overset{-}{\frac{13}{16}}\right)}, \cdots\cdots \left(\overset{-}{\frac{15}{16}}, \overset{-}{1}\right).$$

4. Then considering the same intervals, roots to the approximation of $\frac{1}{16^2}$, lie in the intervals marked.

$$\left\{\begin{array}{lllll} \left(\frac{\overset{-}{1}}{16}, \frac{\overset{-}{2}}{16}\right) \cdots & \cdots & \cdots & \cdots & \cdots \left(\frac{1\overset{-}{+}\frac{15}{16}}{16}, \frac{\overset{-}{2}}{16}\right) \\ \left(\frac{\overset{-}{14}}{16}, \frac{\overset{-}{15}}{16}\right) \cdots & \multicolumn{2}{l}{\overbrace{\left(\frac{14\overset{-}{+}\frac{6}{16}}{16}, \frac{14\overset{+}{+}\frac{7}{16}}{16}\right)} \cdots} & & \overbrace{\left(\frac{14\overset{+}{+}\frac{9}{16}}{16}, \frac{14\overset{-}{+}\frac{10}{16}}{16}\right)} \end{array}\right\}$$

$$\left\{\begin{array}{lllll} \left(\frac{\overset{-}{2}}{16}, \frac{\overset{-}{3}}{16}\right) \cdots & \multicolumn{2}{l}{\overbrace{\left(\frac{2\overset{-}{+}\frac{2}{16}}{16}, \frac{2\overset{+}{+}\frac{3}{16}}{16}\right)} \cdots} & & \overbrace{\left(\frac{2\overset{+}{+}\frac{15}{16}}{16}, \frac{\overset{-}{3}}{16}\right)} \\ \left(\frac{\overset{-}{13}}{16}, \frac{\overset{-}{14}}{16}\right) \cdots & \cdots & \cdots & \cdots & \cdots \left(\frac{13\overset{-}{+}\frac{15}{16}}{16}, \frac{\overset{-}{14}}{16}\right) \end{array}\right\}$$

$$\left\{\begin{array}{lllll} \left(\frac{\overset{-}{3}}{16}, \frac{\overset{+}{4}}{16}\right) \cdots & \cdots & \cdots & \cdots & \cdots \overbrace{\left(\frac{3\overset{-}{+}\frac{15}{16}}{16}, \frac{\overset{+}{4}}{16}\right)} \\ \left(\frac{\overset{+}{12}}{16}, \frac{\overset{-}{13}}{16}\right) \cdots & \cdots & \cdots & \cdots & \cdots \overbrace{\left(\frac{12\overset{+}{+}\frac{15}{16}}{16}, \frac{\overset{-}{13}}{16}\right)} \end{array}\right\}$$

$$\left\{\begin{array}{lllll} \left(\frac{\overset{+}{4}}{16}, \frac{\overset{+}{5}}{16}\right) \cdots & \cdots & \cdots & \cdots & \cdots \left(\frac{4\overset{+}{+}\frac{15}{16}}{16}, \frac{\overset{+}{5}}{16}\right) \\ \left(\frac{\overset{+}{11}}{16}, \frac{\overset{+}{12}}{16}\right) \cdots & \multicolumn{2}{l}{\overbrace{\left(\frac{11\overset{+}{+}\frac{1}{16}}{16}, \frac{11\overset{-}{+}\frac{2}{16}}{16}\right)} \cdots} & & \overbrace{\left(\frac{11\overset{-}{+}\frac{15}{16}}{16}, \frac{\overset{+}{12}}{16}\right)} \end{array}\right\}$$

$$\left\{\begin{array}{lllll} \left(\frac{\overset{+}{5}}{16}, \frac{\overset{+}{6}}{16}\right) \cdots & \multicolumn{2}{l}{\overbrace{\left(\frac{5\overset{+}{+}\frac{2}{16}}{16}, \frac{5\overset{-}{+}\frac{3}{16}}{16}\right)} \cdots} & & \overbrace{\left(\frac{5\overset{-}{+}\frac{13}{16}}{16}, \frac{5\overset{+}{+}\frac{14}{16}}{16}\right)} \\ \left(\frac{\overset{+}{10}}{16}, \frac{\overset{+}{11}}{16}\right) \cdots & \cdots & \cdots & \cdots & \cdots \left(\frac{10\overset{+}{+}\frac{15}{16}}{16}, \frac{\overset{+}{11}}{16}\right) \end{array}\right\}$$

$$\left\{\begin{array}{llll}
\left(\frac{\overset{+}{6}}{16},\frac{\overset{+}{7}}{16}\right)\cdots & \cdots\quad\cdots & \cdots & \cdots\left(\frac{6\overset{+}{+}\frac{15}{16}}{16},\frac{\overset{+}{7}}{16}\right)\\[2ex]
\left(\frac{\overset{+}{9}}{16},\frac{\overset{+}{10}}{16}\right)\cdots & \overbrace{\left(\frac{9\overset{+}{+}\frac{3}{16}}{16},\ \frac{9\overset{-}{+}\frac{4}{16}}{16}\right)} & \cdots & \overbrace{\left(\frac{9\overset{-}{+}\frac{12}{16}}{16},\ \frac{9\overset{+}{+}\frac{13}{16}}{16}\right)}
\end{array}\right\}$$

$$\left\{\begin{array}{llll}
\left(\frac{\overset{+}{7}}{16},\frac{\overset{+}{8}}{16}\right)\cdots & \overbrace{\left(\frac{7\overset{+}{+}\frac{6}{16}}{16},\ \frac{7\overset{-}{+}\frac{7}{16}}{16}\right)} & \cdots & \overbrace{\left(\frac{7\overset{-}{+}\frac{9}{16}}{16},\ \frac{7\overset{+}{+}\frac{10}{16}}{16}\right)}\\[2ex]
\left(\frac{\overset{+}{8}}{16},\frac{\overset{+}{9}}{16}\right)\cdots & \cdots\quad\cdots & \cdots & \cdots\left(\frac{8\overset{+}{+}\frac{15}{16}}{16},\frac{\overset{+}{9}}{16}\right)
\end{array}\right\}$$

5. Next I consider the interval $(0, \frac{1}{16})$ more closely and separate the roots therein by the intervals marked in the following, the roots being approximated to $\frac{1}{16^4}$.

$$\left(\frac{\overset{-}{1}}{16},\ \frac{\overset{-}{15}}{16^2}\right).$$

$$\left(\frac{\overset{-}{15}}{16^2},\frac{\overset{-}{14}}{16^2}\right)\cdots\cdots\left(\frac{14}{16^2}+\overbrace{\frac{\overset{-}{4}}{16^3}+\frac{5}{16^4},\frac{14}{16^2}+\frac{\overset{+}{4}}{16^3}}+\frac{6}{16^4}\right)\cdots\cdots$$

$$\left(\frac{14}{16^2}+\overbrace{\frac{\overset{+}{4}}{16^3}+\frac{11}{16^4},\frac{14}{16^2}+\frac{\overset{-}{4}}{16^3}}+\frac{12}{16^4}\right).$$

$$\left(\frac{14}{16^2}+\overbrace{\frac{\overset{-}{6}}{16^3}+\frac{3}{16^4},\frac{14}{16^2}+\frac{\overset{+}{6}}{16^3}}+\frac{4}{16^4}\right)\cdots\cdots$$

$$\left(\frac{14}{16^2}+\frac{6}{16^3}\overbrace{\overset{+}{+}\frac{13}{16^4},\frac{14}{16^2}+\frac{\overset{-}{6}}{16^3}}+\frac{14}{16^4}\right).$$

$$\left(\frac{14}{16^2}+\overbrace{\frac{\overset{-}{8}}{16^3}+\frac{2}{16^4},\frac{14}{16^2}+\frac{\overset{+}{8}}{16^3}}+\frac{3}{16^4}\right)\cdots\cdots$$

$$\left(\frac{14}{16^2}+\overbrace{\frac{\overset{+}{8}}{16^3}+\frac{12}{16^4},\frac{14}{16^2}+\frac{\overset{-}{8}}{16^3}}+\frac{13}{16^4}\right).$$

$$\left(\frac{14}{16^2}+\overbrace{\frac{\overset{-}{10}}{16^3}+\frac{4}{16^4},\frac{14}{16^2}+\frac{\overset{+}{10}}{16^3}}+\frac{5}{16^4}\right)\ldots\ldots$$

$$\left(\frac{14}{16^2}+\frac{10}{16^3}\overbrace{\overset{+}{+}\frac{10}{16^4},\frac{14}{16^2}+\frac{\overset{-}{10}}{16^3}}+\frac{11}{16^4}\right).$$

$$\left(\frac{\bar{14}}{16^2},\frac{\bar{13}}{16^2}\right)$$

$$\left(\frac{\bar{13}}{16^2},\frac{\bar{12}}{16^2}\right)\qquad\left(\frac{12}{16^2}\overbrace{\overset{-}{+}\frac{6}{16^4},\frac{12}{16^2}\overset{+}{+}}\frac{7}{16^4}\right)\ldots\ldots$$

$$\left(\frac{12}{16^2}\overbrace{\overset{+}{+}\frac{10}{16^4},\frac{12}{16^4}\overset{-}{+}}\frac{11}{16^4}\right).$$

$$\left(\frac{12}{16^2}+\overbrace{\frac{\overset{-}{2}}{16^3}+\frac{1}{16^4},\frac{12}{16^2}+\frac{\overset{+}{2}}{16^3}}+\frac{2}{16^4}\right)\ldots\ldots$$

$$\left(\frac{12}{16^2}+\frac{2}{16^3}\overbrace{\overset{-}{+}\frac{15}{16^4},\frac{12}{16^2}\overset{+}{+}}\frac{3}{16^3}\right).$$

$$\left(\frac{12}{16^2}+\frac{3}{16^3}\overbrace{\overset{-}{+}\frac{15}{16^4},\frac{12}{16^2}\overset{+}{+}}\frac{4}{16^3}\right).$$

$$\left(\frac{12}{16^2}+\frac{5}{16^3}\overbrace{\overset{+}{+}\frac{2}{16^4},\frac{12}{16^2}+\frac{\overset{-}{5}}{16^3}}+\frac{8}{16^4}\right)\ldots\ldots$$

$$\left(\frac{12}{16^2}+\frac{5}{16^3}\overbrace{\overset{+}{+}\frac{13}{16^4},\frac{12}{16^2}+\frac{\overset{-}{5}}{16^3}}+\frac{14}{16^4}\right).$$

$$\left(\frac{12}{16^2}+\frac{7}{16^3}\overbrace{\overset{+}{+}\frac{3}{16^4},\frac{12}{16^2}+\frac{\overset{-}{7}}{16^3}}+\frac{4}{16^4}\right)\ldots\ldots$$

$$\left(\frac{12}{16^2}+\overbrace{\frac{\overset{-}{7}}{16^3}+\frac{12}{16^4},\frac{12}{16^2}+\frac{\overset{+}{7}}{16^3}}+\frac{13}{16^4}\right).$$

$$\left(\frac{12}{16^2}+\frac{9}{16^3}\overbrace{\overset{+}{+}\frac{3}{16^4},\frac{12}{16^2}+\frac{\overset{-}{9}}{16^3}}+\frac{4}{16^4}\right)\ldots\ldots$$

$$\left(\frac{12}{16^2}+\overbrace{\frac{\overset{-}{9}}{16^3}+\frac{13}{16^4},\frac{12}{16^2}\overset{+}{+}\frac{9}{16^3}}+\frac{14}{16^4}\right).$$

$$\left(\frac{12}{16^2}+\overbrace{\frac{\overset{+}{11}}{16^3}+\frac{1}{16^4},\frac{12}{16^2}+\frac{\overset{-}{11}}{16^3}}+\frac{2}{16^4}\right).$$

$$\left(\frac{12}{16^2}\overbrace{\overset{-}{+}\frac{12}{16^3},\frac{12}{16^2}+\frac{12}{16^3}\overset{+}{+}}\frac{1}{16^4}\right)\ldots\ldots$$

$$\left(\frac{12}{16^2}+\frac{12}{16^3}\overbrace{\overset{+}{+}\frac{14}{16^4},\frac{12}{16^2}+\frac{\overset{-}{12}}{16^3}}+\frac{15}{16^4}\right).$$

$$\left(\frac{12}{16^2}+\overbrace{\frac{\overset{-}{14}}{16^3}+\frac{6}{16^4},\frac{12}{16^2}+\frac{14}{16^3}\overset{+}{+}}\frac{7}{16^4}\right)\ldots\ldots$$

$$\left(\frac{12}{16^2}+\overbrace{\frac{\overset{+}{14}}{16^3}+\frac{8}{16^4},\frac{12}{16^2}+\frac{\overset{-}{14}}{16^3}}+\frac{9}{16^4}\right).$$

$$\left(\frac{\bar{12}}{16^2},\frac{\bar{11}}{16^2}\right)\quad\left(\frac{11}{16^2}\overbrace{\overset{-}{+}\frac{4}{16^4},\frac{11}{16^2}\overset{+}{+}}\frac{5}{16^4}\right)\ldots\ldots$$

$$\left(\frac{11}{16^2}\overbrace{\overset{+}{+}\frac{9}{16^4},\frac{11}{16^2}\overset{-}{+}}\frac{10}{16^4}\right).$$

$$\left(\frac{\bar{11}}{16^2}+\frac{\bar{10}}{16^2}\right)\quad\left(\frac{10}{16^2}\overbrace{\overset{-}{+}\frac{1}{16^4},\frac{10}{16^2}\overset{+}{+}}\frac{2}{16^4}\right)\quad\ldots\ldots$$

$$\left(\frac{10}{16^2}\overbrace{\overset{+}{+}\frac{15}{16^4},\frac{10}{16^2}\overset{-}{+}}\frac{1}{16^3}\right).$$

$$\left(\frac{10}{16^2}+\overbrace{\frac{\overset{-}{1}}{16^3}+\frac{14}{16^4},\frac{10}{16^2}+\frac{\overset{+}{1}}{16^3}}+\frac{15}{16^4}\right).$$

$$\left(\frac{10}{16^2}+\overbrace{\frac{\overset{+}{3}}{16^3}+\frac{4}{16^4},\frac{10}{16^2}+\frac{\overset{-}{3}}{16^3}}+\frac{5}{16^4}\right)\ldots\ldots$$

$$\left(\frac{10}{16^2}+\overbrace{\frac{\overset{-}{3}}{16^3}+\frac{10}{16^4},\frac{10}{16^2}+\frac{\overset{+}{3}}{16^3}}+\frac{11}{16^4}\right).$$

$$\left(\frac{10}{16^2}+\overbrace{\frac{\overset{+}{13}}{16^3}+\frac{2}{16^4},\frac{10}{16^2}+\frac{\overset{-}{13}}{16^3}}+\frac{8}{16^4}\right)\ldots\ldots$$

$$\left(\frac{10}{16^2}+\overbrace{\frac{\overset{-}{13}}{16^3}+\frac{15}{16^4},\frac{10}{16^2}\overset{+}{+}}\frac{14}{16^3}\right).$$

$$\left(\frac{10}{16^2}\overbrace{\overset{+}{+}\frac{14}{16^3}+\frac{14}{16^4},\frac{10}{16^2}+\frac{\overset{-}{14}}{16^3}}+\frac{15}{16^4}\right).$$

$$\left(\frac{\bar{10}}{16^2},\frac{\bar{9}}{16^2}\right)\quad\left(\frac{9}{16^2}\overbrace{\overset{-}{+}\frac{1}{16^4},\frac{9}{16^2}\overset{+}{+}}\frac{2}{16^4}\right)\cdots\cdots$$

$$\left(\frac{9}{16^2}\overbrace{\overset{+}{+}\frac{12}{16^4},\frac{9}{16^2}\overset{-}{+}}\frac{13}{16^4}\right).$$

$$\left(\frac{9}{16^2}+\overbrace{\frac{\bar{14}}{16^3}+\frac{7}{16^4},\frac{9}{16^2}+\frac{\overset{+}{14}}{16^3}}+\frac{8}{16^4}\right)\cdots\cdots$$

$$\left(\frac{9}{86^2}+\overbrace{\frac{\overset{+}{14}}{16^3}+\frac{9}{16^4},\frac{9}{16^2}+\frac{\bar{14}}{16^3}}+\frac{10}{16^4}\right).$$

$$\left(\frac{\bar{9}}{16^2},\frac{\bar{8}}{16^2}\right)\quad\left(\frac{8}{16^2}\overbrace{\overset{-}{+}\frac{1}{16^4},\frac{8}{16^2}\overset{+}{+}}\frac{2}{16^4}\right)\cdots\cdots$$

$$\left(\frac{8}{16^2}\overbrace{\overset{+}{+}\frac{14}{16^4},\frac{8}{16^2}}\overbrace{\overset{-}{+}\frac{15}{16^4},\frac{8}{16^2}\overset{+}{+}}\frac{1}{16^3}\right).$$

$$\left(\frac{8}{16^2}+\overbrace{\frac{\overset{+}{1}}{16^3},\frac{8}{16^2}+\frac{\bar{1}}{16^2}}+\frac{1}{16^4}\right)\cdots\cdots$$

$$\left(\frac{8}{16^2}+\frac{1}{16^3}\overbrace{\overset{-}{+}\frac{13}{16^4},\frac{8}{16^2}+\frac{1}{16^3}\overset{+}{+}}\frac{14}{16^4}\right).$$

$$\left(\frac{8}{16^2}+\overbrace{\frac{\overset{+}{3}}{16^3}+\frac{6}{16^4},\frac{8}{16^2}+\frac{\bar{3}}{16^3}}+\frac{7}{16^4}\right)\cdots\cdots$$

$$\left(\frac{8}{16^2}+\overbrace{\frac{\bar{3}}{16^3}+\frac{9}{16^4}\cdot\frac{8}{16^2}+\frac{\overset{+}{3}}{16^3}}+\frac{10}{16^4}\right).$$

$$\left(\frac{8}{16^2}+\overbrace{\frac{\overset{+}{13}}{16^3}+\frac{4}{16^4},\frac{8}{16^2}+\frac{\overset{-}{13}}{16^3}}+\frac{5}{16^4}\right)\ldots\ldots$$

$$\left(\frac{8}{16^2}+\overbrace{\frac{\overset{-}{13}}{16^3}+\frac{13}{16^4},\frac{8}{16^2}+\frac{\overset{+}{13}}{16^3}}+\frac{14}{16^4}\right).$$

$$\left(\frac{8}{16^2}\overbrace{\overset{+}{+}\frac{15}{16^3},\frac{8}{16^2}+\frac{\overset{-}{15}}{16^3}}+\frac{1}{16^4}\right).$$

$$\left(\frac{\bar{8}}{16^2},\frac{\bar{7}}{16^2}\right)\qquad\left(\frac{7}{16^2}\overbrace{\overset{-}{+}\frac{2}{16^4},\frac{7}{16^2}\overset{+}{+}}\frac{3}{16^4}\right)\ldots\ldots$$

$$\left(\frac{7}{16^2}\overbrace{\overset{+}{+}\frac{11}{16^4},\frac{7}{16^2}\overset{-}{+}}\frac{12}{16^4}\right).$$

$$\left(\frac{7}{16^2}+\overbrace{\frac{\overset{-}{14}}{16^3}+\frac{6}{16^4},\frac{7}{16^2}\frac{\overset{+}{14}}{16^3}}+\frac{7}{16^4}\right)\ldots\ldots$$

$$\left(\frac{7}{16^2}+\overbrace{\frac{\overset{+}{14}}{16^3}+\frac{11}{16^4},\frac{7}{16^2}+\frac{\overset{-}{14}}{16^3}}+\frac{12}{16^4}\right).$$

$$\left(\frac{\bar{7}}{16^2},\frac{\bar{6}}{16^2}\right)\qquad\left(\frac{6}{16^2}\overbrace{\overset{-}{+}\frac{3}{16^4},\frac{6}{16^2}\overset{+}{+}}\frac{4}{16^4}\right)\ldots\ldots$$

$$\left(\frac{6}{16^2}\overbrace{\overset{+}{+}\frac{13}{16^4},\frac{6}{16^2}\overset{-}{+}}\frac{14}{16^4}\right).$$

$$\left(\frac{6}{16^2}+\overbrace{\frac{\bar{1}}{16^3}+\frac{15}{16^4},\frac{6}{16^2}\overset{+}{+}}\frac{2}{16^3}\right).$$

$$\left(\frac{6}{16^2}+\overbrace{\frac{\overset{+}{3}}{16^3}+\frac{2}{16^4},\frac{6}{16^2}+\frac{\overset{-}{3}}{16^3}}+\frac{3}{16^4}\right)\ldots\ldots$$

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$$\left(\frac{6}{16^2}+\overbrace{\frac{\overset{+}{13}}{16^3}+\frac{2}{16^4},\frac{6}{16^2}+\frac{\overset{-}{13}}{16^3}}+\frac{3}{16^4}\right)\ldots\ldots$$

$$\left(\frac{6}{16^2}+\overbrace{\frac{\overset{-}{13}}{16^3}+\frac{14}{16^4},\frac{6}{16^2}+\frac{\overset{+}{13}}{16^3}}+\frac{15}{16^4}\right).$$

$$\left(\frac{6}{16^2}+\overbrace{\frac{\overset{+}{14}}{16^3}+\frac{15}{16^4},\frac{6}{16^2}}\overset{-}{+}\frac{15}{16^3}\right).$$

$$\left(\frac{\overset{-}{6}}{16^2},\frac{\overset{-}{5}}{16^2}\right)$$

$$\left(\frac{\overset{-}{5}}{16^2},\frac{\overset{-}{4}}{16^2}\right)\qquad\left(\frac{4}{16^2}+\overbrace{\frac{\overset{-}{2}}{16^3}+\frac{5}{16^4},\frac{4}{16^2}+\frac{\overset{+}{2}}{16^3}}+\frac{6}{16^4}\right)\ldots\ldots$$

$$\left(\frac{4}{16^2}+\overbrace{\frac{\overset{+}{2}}{16^3}+\frac{11}{16^4},\frac{4}{16^2}+\frac{\overset{-}{2}}{16^3}}+\frac{12}{16^4}\right).$$

$$\left(\frac{4}{16^2}+\overbrace{\frac{\overset{-}{4}}{16^3}+\frac{1}{16^4},\frac{4}{16^2}+\frac{\overset{+}{4}}{16^3}}+\frac{2}{16^4}\right)\ldots\ldots$$

$$\left(\frac{4}{16^2}+\overbrace{\frac{\overset{+}{4}}{16^3}+\frac{15}{16^4},\frac{4}{16^2}}\overset{-}{+}\frac{5}{16^3}\right).$$

$$\left(\frac{4}{16^2}+\frac{5}{16^3}\overset{-}{+}\overbrace{\frac{14}{16^4},\frac{4}{16^2}+\frac{\overset{+}{5}}{16^3}}+\frac{15}{16^4}\right).$$

$$\left(\frac{4}{16^2}+\frac{7}{16^3}\overbrace{\overset{+}{+}\frac{1}{16^4},\frac{4}{16^2}+\frac{\overset{-}{7}}{16^3}}+\frac{2}{16^4}\right)\ldots\ldots$$

$$\left(\frac{4}{16^2}+\frac{7}{16^3}\overbrace{\overset{-}{+}\frac{13}{16^4},\frac{4}{16^2}+\frac{\overset{+}{7}}{16^3}}+\frac{14}{16^4}\right).$$

$$\left(\frac{4}{16^2}+\frac{9}{16^3}\overbrace{\overset{+}{+}\frac{2}{16^4},\frac{4}{16^2}+\frac{\overset{-}{9}}{16^3}}+\frac{3}{16^4}\right)\ldots\ldots$$

$$\left(\frac{4}{16^2}+\frac{9}{16^3}\overbrace{\overset{-}{+}\frac{14}{16^4},\frac{4}{16^2}+\frac{\overset{+}{9}}{16^3}}+\frac{15}{16^4}\right).$$

$$\left(\frac{4}{16^2}\overbrace{+\frac{\overset{+}{11}}{16^3}+\frac{1}{16^4},\frac{4}{16^2}+\frac{\overset{-}{11}}{16^3}}+\frac{2}{16^4}\right).$$

$$\left(\frac{4}{16^2}\overbrace{\overset{-}{+}\frac{12}{16^3},\frac{4}{16^2}+\frac{\overset{+}{12}}{16^3}}+\frac{1}{16^4}\right)\ldots\ldots$$

$$\left(\frac{4}{16^2}\overbrace{+\frac{\overset{+}{12}}{16^3}+\frac{14}{16^4},\frac{4}{16^2}+\frac{\overset{-}{12}}{16^3}}+\frac{15}{16^4}\right).$$

$$\left(\frac{4}{16^2}\overbrace{+\frac{\overset{-}{14}}{16^3}+\frac{3}{16^4},\frac{4}{16^2}+\frac{\overset{+}{14}}{16^3}}+\frac{4}{16^4}\right)\ldots\ldots$$

$$\left(\frac{4}{16^2}\overbrace{+\frac{\overset{+}{14}}{16^3}+\frac{11}{16^4},\frac{4}{16^2}+\frac{\overset{-}{14}}{16^3}}+\frac{12}{16^4}\right).$$

$$\left(\frac{\overset{-}{4}}{16^2},\frac{\overset{-}{3}}{16^3}\right)$$

$$\left(\frac{\overset{+}{3}}{16^2},\frac{\overset{-}{2}}{16^3}\right)\qquad\left(\frac{2}{16^2}\overbrace{+\frac{\overset{-}{8}}{16^3}+\frac{6}{16^4},\frac{2}{16^2}+\frac{\overset{+}{8}}{16^3}}+\frac{7}{16^4}\right)\ldots\ldots$$

$$\left(\frac{2}{16^2}+\frac{8}{16^3}+\overbrace{\frac{\overset{+}{9}}{16^4},\frac{2}{16^2}+\frac{\bar{8}}{16^3}}+\frac{10}{16^4}\right).$$

$$\left(\frac{\bar{2}}{16^2},\frac{\bar{1}}{16^2}\right)$$

$$\left(\bar{0},\frac{\bar{1}}{16^2}\right)\qquad\left(\bar{0},\frac{8}{16^3}+\frac{8}{16^4}\right)\left(\bar{0},\frac{8}{16^3}+\frac{\bar{8}}{16^4}+\frac{8}{16^5}\right)\ldots\ldots\ldots$$

Thus the value of $W(\theta)$ continually increases and this fact shows that there may be roots in the interval $\left(0, \frac{1}{16^2}\right)$.

$$\left(\bar{0},\frac{\bar{1}}{16^3}\right)\left(\bar{0},\frac{8}{16^4}+\frac{8}{16^5}\right)\left(\bar{0},\frac{8}{16^4}+\frac{\bar{8}}{16^5}+\frac{8}{16^6}\right)\ldots\ldots$$

and so on, so that in general we have,

$$\left(\bar{0},\frac{\bar{1}}{16^k}\right)\left(\bar{0},\frac{8}{16^{k+1}}+\frac{8}{16^{k+2}}\right)\left(\bar{0},\frac{8}{16^{k+1}}+\frac{\bar{8}}{16^{k+2}}+\frac{8}{16^{k+3}}\right)\ldots\ldots$$

6. To show that in this general case there exists an interval $\left(0,\frac{1}{16^l}\right)$ in which there seem to be no zeroes of $W(\theta)$.

Proof :—

Considering the interval $\left(\frac{1}{16^3}, \frac{1}{16^4}\right)$ we find that the greatest possible value which the expression $\sum\limits_0^\infty \frac{\sin(16^n\theta\pi)}{2^n}$ can attain, can be obtained only by the following value of θ,

$$\left\{\frac{8}{16^4}+\frac{8}{16^5}+\frac{8}{16^6}\ldots\ldots\infty\right\} \textit{i.e.,}\ \frac{8}{16^3.15}.$$

Whence, $\displaystyle\sum_0^\infty \frac{\sin(16^n\theta\pi)}{2^n}$

$$=\sin\left(\frac{8\pi}{16^3 15}\right)+\frac{1}{2}\sin\left(\frac{16.\ 8\pi}{16^3.15}\right)+\frac{1}{2^2}\left(\frac{16^2.8\pi}{16^3.15}\right)+\ldots$$

$$<\frac{8\pi}{16^3.15}\left\{1+8+8^2\right\}+\frac{1}{2^3}\cdot\frac{16^3.7\pi}{16^3\ 15}\left\{1+\frac{1}{2}+\frac{1}{2^2}+\ldots\infty\right\}.$$

$$<\frac{7752\pi}{16^3.15}.$$

But $$\frac{7752\pi}{16^3.15}<\frac{2}{\pi}.$$

$\therefore$ $W(\theta)$ can never be positive in the interval $\left(\frac{1}{16^3},\frac{1}{16^4}\right)$ and consequently in the interval $\left(0,\frac{1}{16^3}\right)$ and hence there seems to be no possibility of there being roots of $W(\theta)$ in it.

Or more generally taking the interval $\left(\frac{1}{16^k},\frac{1}{16^{k+1}}\right)$ where k is a positive integer greater than 2, we have the following value of θ which gives the greatest positive value to the expression $\sum_0^\infty \frac{\sin(16^n\theta\pi)}{2^n}$ in this interval,

$$\left\{\frac{8}{16^{k+1}}+\frac{8}{16^{k+2}}+\frac{8}{16^{k+3}}+\ldots\infty\right\}\ i.e.,\ \frac{8}{16^k.15}.$$

Whence, $$\sum_0^\infty \frac{\sin(16^n\theta\pi)}{2^n}$$

$$=\sin\left(\frac{8\pi}{16^k.15}\right)+\frac{1}{2}\sin\left(\frac{16.8\pi}{16^k.15}\right)+\frac{1}{2^2}\sin\left(\frac{16^2.8\pi}{16^k.15}\right)+\ldots$$

$$+\frac{1}{2^k}\sin\left(\frac{16^k.7\pi}{16^k.15}\right)+\frac{1}{2^{k+1}}\sin\left(\frac{16^k.7\pi}{16^k.15}\right)+\ldots$$

$$<\frac{8\pi}{16^k.15}\{1+8+8^2+\ldots.+8^{k-1}\}+\frac{1}{2^{k-1}}\cdot\frac{16^k.7\pi}{16^k.15}.$$

$$<\frac{8(8^k-1)\pi}{16^k.15.7}+\frac{16^k.7\pi}{2^{k-1}16^k.15}.$$

Now here k can be so chosen that the quantity $\left\{\frac{8(8^k-1)\pi}{16^k.15.7} + \frac{16^k.7\pi}{2^{k-1}16^k.15}\right\}$ can be made less than K where K is small, and so that an interval can be obtained which seems not to contain the zeroes of $\sum_0^\infty \frac{\sin(16^n\theta\pi)}{2^n} - \mathrm{K} \equiv \overline{\mathrm{W}}(\theta)$ say, and we notice that as K gets smaller and smaller, k gets larger and larger and consequently the interval $\left(\frac{1}{16^{k+1}}, \frac{1}{16^k}\right)$ gets nearer and nearer to 0, and ultimately when K is equal to zero, we find that the zeroes of $\sum_0^\infty \frac{\sin(16^n\theta\pi)}{2^n}$ lie in every neighbourhood of 0, ever so small and that 0 becomes the limiting point of the zeroes of $\sum_0^\infty \frac{\sin(16^n\theta\pi}{2^n}$.

7. The same results can be obtained for the values of θ for $(\frac{1}{16}, \frac{2}{16}, \dots \frac{15}{16}, 1)$ which become the limiting points of the zeroes of $\sum_0^\infty \frac{\sin(16^n\theta\pi)}{2^n}$ in the case when the constant K vanishes; the values of θ, $(0, \frac{1}{16}, \frac{2}{16}, \dots \frac{15}{16}, 1)$ have been proved to be the limiting points of the zeroes of $\sum_0^\infty \frac{\sin(16^n\theta\pi)}{2^n}$ in his paper* by Bhola Nath Mukhopadhyay so that in the case of $\mathrm{W}(\theta) \equiv \sum_0^\infty \frac{\sin(16^n\theta\pi)}{2n} - \frac{2}{\pi} = 0$, we have obtained zeroes of $\mathrm{W}(\theta)$ in the interval (0, 1) of such a type that there can be the least value of θ. Similarly considering the intervals near 1 we can show that there exists the greatest value of θ, so that the zeroes *probably* form an enumerable set. The principal value $\theta_1(h)$ will be single-valued, continuous and nowhere differentiable.

Just in the same way the general non-differentiable function $f(x) \equiv \sum_1^\infty a^n \sin(b^n x) = 0$ where $0 < a < 1$ and b is an even integer and that $ab > 1 + \frac{3\pi}{2}$ can be treated and the corresponding values of θ for a particular value of h can be determined.

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* *Bulletin of the Calcutta Mathematical Society*, Vol. XXII, pp. 108-114.

In this case the θ equation becomes,

$$\sum_{0}^{\infty} a^{n} \sin (b^{n}\theta\pi)-\frac{2}{\pi}=0$$

and this can be solved to give values of θ for $h=\frac{\pi}{b}$.

§ 2.

Behaviour of $\theta_1(h)$ with vanging h.

8. Under this head I will investigate how $\theta_1(h)$ behaves as h varies. Before discussing this question, I will first give zeroes of $W(\theta)$ in the interval $(\frac{15}{16}, 1)$.

The roots θ, to the approximation of $\frac{1}{16^6}$, are shown to lie in the intervals marked in the following,

$$\left(\frac{15}{16}, 1\right)$$

$$\left(\frac{15}{16}+\frac{8}{16^3}+\overbrace{\frac{8}{16^4}+\frac{8}{16^5},\frac{15}{16}+\frac{8}{16^3}+\frac{8}{16^4}}+\overbrace{\frac{8}{16^5}+\frac{8}{16^6},\frac{15}{16}}+\frac{1}{16^2}\right).$$

Similarly as before, the values of θ lying between the intervals,

$$\left\{\left(\frac{15}{16}+\frac{1}{16^2}\right),\left(\frac{15}{16}+\frac{3}{16^2}\right)\right\} \left\{\left(\frac{15}{16}+\frac{3}{16^2}\right),\left(\frac{15}{16}+\frac{5}{16^2}\right)\right\} \cdots$$

$$\cdots \quad \cdots \quad \left\{\left(\frac{15}{16}+\frac{13}{16^2}\right),\left(\frac{15}{16}+\frac{15}{16^2}\right)\right\},$$

which give the greatest positive value to the expression $\sum_{0}^{\infty}\frac{\sin(16^{n}\theta\pi)}{2^{n}}$ in the corresponding intervals are,

$$\left\{\frac{15}{16}+\frac{2}{16^2}+\frac{8}{16^3}+\frac{8}{16^4}+\ldots\infty\right\} \left\{\frac{15}{16}+\frac{4}{16^2}+\frac{8}{16^3}+\frac{8}{16^4}+\ldots\infty\right\} \cdots$$

$$\cdots \cdots \left\{\frac{15}{16}+\frac{14}{16^2}+\frac{8}{16^3}+\frac{8}{16^4}+\cdots\infty\right\}.$$

For such values of θ in the intervals, marked $\}$, we can, by applying the inequality $1 > \frac{\sin\theta}{\theta} > \frac{2}{\pi}$, find the limits between which the greatest positive value G of the expression $\sum_{0}^{\infty} \frac{\sin(16^{n}\theta\pi)}{2^{n}}$ will lie.

$\left(\frac{15}{16}+\frac{1}{16^2}\right)$ $\left(\frac{15}{16}+\frac{3}{16^2}\right)$	For this interval G	$> \cdot 322$	and	$< \cdot 740$
$\left(\frac{15}{16}+\frac{3}{16^2}\right)$ $\left(\frac{15}{16}+\frac{5}{16^2}\right)$	,,	$> \cdot 111$	,,	$< \cdot 590$
$\left(\frac{15}{16}+\frac{5}{16^2}\right)$ $\left(\frac{15}{16}+\frac{7}{16^2}\right)$	,,		,,	$< \cdot 466$
$\left(\frac{15}{16}+\frac{7}{16^2}\right)$ $\left(\frac{15}{16}+\frac{9}{16^2}\right)$	,,		,,	$< \cdot 357$
$\left(\frac{15}{16}+\frac{9}{16^2}\right)$ $\left(\frac{15}{16}+\frac{11}{16^2}\right)$	,,		,,	$< \cdot 458$
$\left(\frac{15}{16}+\frac{11}{16^2}\right)$ $\left(\frac{15}{16}+\frac{13}{16^2}\right)$	,,	$> \cdot 153$	,,	$< \cdot 556$
$\left(\frac{15}{16}+\frac{13}{16^2}\right)$ $\left(\frac{15}{16}+\frac{15}{16^2}\right)$	,,		,,	$< \cdot 659$

9. From the above we can at once say that no root θ of $W(\theta)$ seems to lie between the interval $\left\{\left(\frac{15}{16}+\frac{3}{16^2}\right),\left(\frac{15}{16}+\frac{13}{16^2}\right)\right\}$, but at first sight we cannot say anything definite about the existence of roots in intervals $\left\{\left(\frac{15}{16}+\frac{1}{16^2}\right),\left(\frac{15}{16}+\frac{3}{16^2}\right)\right\}$, $\left\{\left(\frac{15}{16}+\frac{13}{16^2}\right),\left(\frac{15}{16}+\frac{15}{16^3}\right)\right\}$ but most probably there seems to be no root in them.

Next considering the interval $\left\{\left(\frac{15}{16}+\frac{15}{16^2}\right), 1\right\}$ we find that the value of θ which gives the greatest positive value to the expression $\sum_0^\infty \frac{\sin(16^n\theta\pi)}{2^n}$ is $\left\{\frac{15}{16}+\frac{15}{16^2}+\frac{2}{16^3}+\frac{8}{16^4}+\frac{8}{16^6}+\dots\infty\right\}$ and by applying the inequality $1 > \frac{\sin\theta}{\theta} > \frac{2}{\pi}$ we find that $\sum_0^\infty \frac{\sin(16^n\theta\pi)}{2^n}$ is $> \cdot 032$ and $< \cdot 245$ and therefore there seems to be no possible root of $W(\theta)$ in this interval.

10. From the above it also follows that as the constant in $\overline{W}(\theta)$ gets smaller and smaller, the principal value $\theta_1(h)$ comes closer and closer to 1 and ultimately when the constant vanishes it actually becomes 1; while if the constant gets larger, $\theta_1(h)$ recedes from 1 and there can exist a value of the constant for which there may be no zeroes of $\overline{W}(\theta)$.

11. Next in order to consider the above question I obtain different equations giving values of θ for corresponding values of h, (h being first increasing and then decreasing).

For h increasing,

$h=\frac{\pi}{16}$ θ Equation: $\sum_0^\infty \frac{\sin(16^n\theta\pi)}{2^n} - \frac{2}{\pi} = 0$

$h=\frac{2\pi}{16}$ " $\sum_0^\infty \frac{\sin(16^n 2\theta\pi)}{2^n} = 0$

$h=\frac{3\pi}{16}$ " $\sum_0^\infty \frac{\sin(16^n.3\theta\pi)}{2^n} - \frac{2}{3\pi} = 0$

... ...

$h=\frac{15\pi}{16}$ " $\sum_0^\infty \frac{\sin(16^n 15\theta\pi)}{2^n} - \frac{2}{15\pi} = 0$

$h=\pi$ " $\sum_0^\infty \frac{\sin(16^n.16\theta\pi)}{2^n} = 0$

And in general,

$$h=\frac{2m\pi}{16} \qquad \theta \text{ Equation :} \qquad \sum_0^\infty \frac{\sin(16^n 2m\theta\pi)}{2^n}=0$$

$$h=\frac{(2m+1)\pi}{16} \qquad \text{,,} \qquad \sum_0^\infty \frac{\sin(16^n \overline{2m+1}\theta\pi)}{2^n}-\frac{2}{(2m+1)\pi}=0.$$

12. From these equations it follows that as h increases from $\frac{\pi}{16}$ to $\frac{2\pi}{16}$, the principal value $\theta_1(h)$ increases from some value near 1 to 1 and then as h increases from $\frac{2\pi}{16}$ to $\frac{3\pi}{16}$, $\theta_1(h)$ decreases from 1 to a value which is greater than that occupied by it at $h=\frac{\pi}{16}$ and then it again increases to 1 when h increases from $\frac{3\pi}{16}$ to $\frac{4\pi}{16}$ and then again decreases to a value which is still greater than that occupied by it previously, so that in conclusion I may say that $\theta_1(h)$ can be said to be a special kind of periodic or oscillating function of h, the period of the function being $\frac{\pi}{16}$ and the amplitude rapidly increasing with the increase of h.

Now when h becomes very large, $\theta_1(h)$ becomes very nearly equal to 1, and the amplitude of the function becomes nearly equal to the length of the interval and to the small changes in h, $\theta_1(h)$ oscillates through the small length of the interval.

Also we notice that $\theta_1(\pi)=1$.

For h decreasing,

$$h=\frac{\pi}{16} \qquad \theta \text{ Equation ;} \qquad \sum_0^\infty \frac{\sin(16^n\theta\pi)}{2^n}-\frac{2}{\pi}=0$$

$$h=\frac{15}{16}\cdot\frac{\pi}{16} \qquad \theta \text{ Equation : } \sum_0^\infty \frac{\sin(16^n.\frac{15}{16}\theta\pi)}{2^n}-\frac{1013\cdot47}{465\pi}=0$$

$$h=\frac{14}{16}\cdot\frac{\pi}{16} \qquad \text{''} \qquad \sum_0^\infty \frac{\sin(16^n.\frac{14}{16}\theta\pi)}{2^n}-\frac{15\cdot39}{7\pi}=0$$

$$h=\frac{1}{2}\cdot\frac{\pi}{16} \qquad \text{''} \qquad \sum_0^\infty \frac{\sin(16^n.\frac{1}{2}\theta\pi)}{2^n}-\frac{2}{\pi}=0$$

$$h=\frac{1}{4}\cdot\frac{\pi}{16} \qquad \text{''} \qquad \sum_0^\infty \frac{\sin(16^n.\frac{1}{4}\theta\pi)}{2^n}-\frac{36\cdot31836}{31\pi}=0$$

$$h=\frac{1}{8}\cdot\frac{\pi}{16} \qquad \text{''} \qquad \sum_0^\infty \frac{\sin(16^n.\frac{1}{8}\theta\pi)}{2^n}-\frac{18\cdot87776}{31\pi}=0$$

$$h=\frac{1}{16}\cdot\frac{\pi}{16} \qquad \text{''} \qquad \sum_0^\infty \frac{\sin(16^n.\frac{1}{16}\theta\pi)}{2^n}-\frac{40\cdot52816}{31\pi}=0$$

$$h=\frac{1}{16^2}\cdot\frac{\pi}{16} \qquad \text{''} \qquad \sum_0^\infty \frac{\sin\left(16^n.\ \frac{1}{16^2}\theta\pi\right)}{2^n}-\frac{20\cdot81960}{31\pi}=0$$

$$h=\frac{1}{16^3}\cdot\frac{\pi}{16} \qquad \text{''} \qquad \sum_0^\infty \frac{\sin\left(16^n\cdot\ \frac{1}{16^3}\theta\pi\right)}{2^n}-\frac{11\cdot67956}{31\pi}=0$$

$$h=\frac{1}{16^4}\cdot\frac{\pi}{16} \qquad \text{''} \qquad \sum_0^\infty \frac{\sin\left(16^n\cdot\ \frac{1}{16^4}\theta\pi\right)}{2^n}-\frac{5\cdot83978}{31\pi}=0.$$

so that in general,

$$h=\frac{1}{16^{n-1}}\cdot\frac{\pi}{16} \qquad \text{''} \qquad \sum_0^\infty \frac{\sin\left(16^n.\ \frac{1}{16^{n-1}}\theta\pi\right)}{2^n}-\frac{11\cdot67956}{2^{n-4}\cdot31\pi}=0$$

where n is a positive integer > 3.

13. From the above it follows that as h decreases, the principal value $\theta_1(h)$ increases continually and when h tends to zero, the constant of the θ equation tends to zero and the value $\theta_1(h)$ tends to 1.

§3.

Question of the existence of $\frac{d\theta}{dh}$.

14. Under this head I will consider the question whether $\frac{d\theta}{dh}$ exists for $\theta=0, \frac{1}{16}, \frac{2}{16}, \ldots \frac{15}{16}, 1$, when h is of the form $\frac{2m\pi}{16}$.

By taking h to be $\frac{2m\pi}{16}$ where m is any integer and θ to be $\frac{n}{16}$ where n is equal to $(1, 2, 3, \ldots 16)$, we get $\xi=\frac{2mn\pi}{16^2}$.

Now in the equation

$$f'(\theta h)=\left\{\frac{f(h)-f(0)}{h}\right\},\text{ we have}$$

$$\frac{d}{dh}\left\{\frac{f(h)-f(0)}{h}\right\}=\frac{1}{h}f'(h)-\frac{1}{h^2}f(h)+\frac{1}{h^2}f(0)$$

$$=\frac{1}{h}\sum_1^\infty\frac{\sin(16^n h)}{2^n}+\frac{1}{h^2}\sum_1^\infty\frac{\cos(16^n h)}{2^n.16^n}-\frac{1}{h^2}\cdot\frac{1}{31}$$

$$=\frac{1}{h^2}\sum_1^\infty\frac{\cos(16^n h)}{2^n.16^n}-\frac{1}{h^2}\cdot\frac{1}{31}$$

$$=0,$$

Also, $$f'(\theta h)=\sum_1^\infty\frac{\sin(16^n\cdot\xi)}{2^n}=\omega(\xi).$$

Hence,

$$\operatorname*{Lim}_{\tau\longrightarrow 0}\left\{\frac{\omega(\xi+\tau)-\omega(\xi)}{\tau}\cdot\frac{\tau}{\epsilon}\right\}$$

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exists and equals zero.

Now putting $\xi = \frac{2mn\pi}{16^s}$, we have

$$\omega(\xi) = \sum_1^\infty \frac{\sin\left(16^n . \frac{2mn\pi}{16^s}\right)}{2^n} = \sum_1^\infty \frac{\sin\left(16^{n-1} \frac{2mn\pi}{16}\right)}{2^n}$$

$$= \frac{1}{2} \sum_0^\infty \frac{\sin\left(16^n . \frac{2mn\pi}{16}\right)}{2^n}$$

$$= \frac{1}{2} \, W(\theta) + \frac{1}{\pi}, \text{ where } \theta = \frac{2mn}{16}.$$

But since $\theta = \frac{2mn}{16}$ is not a cusp point* of $W(\theta)$ but only a knot point, *i.e.*, the upper derivatives of W of both the sides are infinite with one sign and the lower derivatives are infinite with the opposite sign, we can only say that most probably $\frac{d\xi}{dh}$ does not exist in this case.

Bull. Cal. Math. Soc., Vol. XXV, No. 1 (1933).

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* See S. K. Bhar's paper "On a continuous function which has no mean differential co-efficient for any value of x." (*Bulletin of the Calcutta Mathematical Society*, Vol. XXIII, pp. 210-230.)

2

"ON SOME SERIES AND INTEGRALS INVOLVING ASSOCIATED LEGENDRE FUNCTIONS"

BY

N. G. SHABDE*

The object of the present paper is to give some series involving P_n^m (cosh σ) or Q_n^m (cosh σ) for unrestricted values of n and m and some integrals involving $P_n^m(z)$. Finally the value of the integral $\int_{-1}^{1} Q_n^2 \,.\, d\mu$ is also noted for unrestricted values of n.

The results given below are believed to be new.

My sincere thanks are due to Prof. Ganesh Prasad for the kind interest he has taken in this paper.

$$(1) \; \frac{1}{\frac{1}{2}} \; P^1_{-\frac{1}{2}}(\cosh\sigma) + \frac{1}{2} \cdot \frac{r}{\frac{1}{2}} \cdot P^1_{\frac{1}{2}}(\cosh\sigma) + \frac{1\cdot 3}{2\cdot 4} \cdot \frac{r^2}{\frac{3}{2}} \cdot P^1_{\frac{3}{2}}(\cosh\sigma) + \ldots \text{to infinity}$$

$$= \frac{2}{\pi\sqrt{e^\sigma - r}} \left[K - \frac{2}{k^2}(E - k'K) \right];$$

the modulus corresponding to K and E being $\sqrt{\dfrac{2\sinh\sigma}{e^\sigma - r}}$ and $r < e^{-\sigma}$.

Proof:

It is known † that

$$P^m_{n-\frac{1}{2}}(\cosh\sigma) = \frac{1}{\pi} \frac{(-1)^m . \, \Pi(n-\frac{1}{2})}{\Pi(n-m-\frac{1}{2})} \int_0^\pi \frac{\cos m\phi \, d\phi}{(\cosh\sigma + \sinh\sigma\cos\phi)^{n+\frac{1}{2}}}$$

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* Received in April, 1933.

† See Hobson's paper "On a Type of Spherical Harmonics of unrestricted Degree, Order and Argument." (Phil. Trans., A, Vol. 187, 1896, see under 'Toroidal Functions').

Therefore, the series has the sum

$$-\frac{1}{\pi}\int_0^{\pi}\frac{\cos\phi\,d\phi}{\{\cosh\sigma-r+\sinh\sigma\cos\phi\}^{\frac{1}{2}}}$$

$$=-\frac{2}{\pi}\int_0^{\pi}\frac{\cos 2\theta.\,d\theta}{\sqrt{e^{\sigma}-r}\ \sqrt{1-k^2\sin^2\theta}},\quad k^2=\frac{2\sinh\sigma}{e^{\sigma}-r}$$

$$=-\frac{2}{\pi}\cdot\frac{1}{\sqrt{e^{\sigma}-r}}\int_0^{\frac{\pi}{2}}\frac{2\cos^2\theta\,d\theta}{\sqrt{1-k^2\sin^2\theta}}+\frac{2}{\pi}\cdot\frac{1}{\sqrt{e^{\sigma}-r}}\int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}$$

$$=\frac{1}{\sqrt{e^{\sigma}-r}}\cdot\frac{2}{\pi}\,\mathrm{K}-\frac{4}{\pi\sqrt{e^{\sigma}-r}}\cdot\int_0^{\mathrm{K}} cn^2 u\,du$$

$$=\frac{1}{\sqrt{e^{\sigma}-r}}\cdot\frac{2}{\pi}\cdot\mathrm{K}-\frac{4}{\pi\sqrt{e^{\sigma}-r}}\cdot\frac{1}{k^2}\Big[\int_0^{\mathrm{K}} dn^2 u.\,du-(k'^2u)_0^{\mathrm{K}}\Big]$$

$$=\frac{2}{\pi}\cdot\frac{1}{\sqrt{e^{\sigma}-r}}\,\mathrm{K}-\frac{4}{\pi\sqrt{e^{\sigma}-r}}\cdot\frac{1}{k^2}\,[\mathrm{E}-k'^2\mathrm{K}].$$

(2) $Q^{\frac{1}{2}}_{\frac{1}{2}}(\cosh\sigma)-\frac{1}{3}\,Q^{\frac{1}{2}}_{\frac{5}{2}}(\cosh\sigma)+\frac{1}{5}\,Q^{\frac{1}{2}}_{\frac{9}{2}}(\cosh\sigma)-+\ldots$to inf.

$$=\frac{i\sqrt{2}\,.\,\Pi(-\frac{1}{2})}{4\sqrt{\sinh\sigma}}\Big[\tan^{-1}\Big\{\coth\frac{\sigma}{2}\Big\}-\tan^{-1}\Big\{\tanh\frac{\sigma}{2}\Big\}\Big]$$

Proof:

It is known * that

$$Q^{m}_{n-\frac{1}{2}}(\cosh\sigma)=(-1)^{m}\,\frac{2^{m}.\,\Pi(m-\frac{1}{2}).\,\Pi(-\frac{1}{2})}{\pi}\int_0^{\pi}\frac{\sinh^{m}\sigma.\,\cos n\phi.\,d\phi}{(2\cosh\sigma-2\cos\phi)^{m+\frac{1}{2}}}$$

Hence

$$Q^{\frac{1}{2}}_{n-\frac{1}{2}}(\cosh\sigma)$$

$$=\frac{i\sqrt{2}\,.\,\Pi(-\frac{1}{2})}{\pi}\int_0^{\pi}\frac{\sqrt{\sinh\sigma}\,.\,\cos n\phi\,d\phi}{2\,(\cosh\sigma-\cos\phi)}$$

---

* Hobson, *l.c.*

Therefore the series in question has the sum

$$\frac{i\sqrt{2}\,\Pi(-\tfrac{1}{2})}{\pi}.\sqrt{\sinh\sigma}.\int_0^{\pi}\left\{\sum_{p=1}^{\infty}\frac{(-1)^{p-1}.\cos(2p-1)\phi}{2p-1}\right\}\frac{d\phi}{2(\cosh\sigma-\cos\phi)}$$

Now the cosine series within the crooked brackets equals $\frac{\pi}{4}$ or $-\frac{\pi}{4}$ according as $\cos\phi$ is + ve or − ve : therefore the sum

$$=\frac{i\sqrt{2}\,.\,\Pi(-\tfrac{1}{2})\,\sqrt{\sinh\sigma}}{\pi}.\frac{\pi}{4}.\left[\int_0^{\frac{\pi}{2}}\frac{d\phi}{2(\cosh\sigma-\cos\phi)}-\int_0^{\frac{\pi}{2}}\frac{d\phi}{2(\cosh\sigma+\cos\phi)}\right]$$

$$=\frac{i\sqrt{2}.\sqrt{\sinh\sigma}\,.\,\Pi(-\tfrac{1}{2})}{4}\left[\frac{1}{\sqrt{\cosh^2\sigma-1}}\left\{\tan^{-1}\sqrt{\frac{\cosh\sigma+1}{\cosh\sigma-1}}-\tan^{-1}\sqrt{\frac{\cosh\sigma-1}{\cosh\sigma+1}}\right\}\right]$$

$$=\frac{i\sqrt{2}.\,\Pi(-\tfrac{1}{2})}{4\sqrt{\sinh\sigma}}\left[\tan^{-1}\left\{\coth\frac{\sigma}{2}\right\}-\tan^{-1}\left\{\tanh\frac{\sigma}{2}\right\}\right]$$

(3) $K_1^{-\frac{1}{2}}(\cosh\sigma)-\tfrac{1}{3}.K_3^{-\frac{1}{2}}(\cosh\sigma)+\tfrac{1}{5}.K_5^{-\frac{1}{2}}(\cosh\sigma)-+-$ to inf.

$$=\frac{\sqrt{\pi}}{2\sqrt{2}}.\frac{\sigma}{\sqrt{\sinh\sigma}},\ 0<\sigma<\frac{\pi}{2}\text{ and taking}$$

$$K_p^{-\frac{1}{2}}(\cosh\sigma)=P_{-\frac{1}{2}+p\,i}^{-\frac{1}{2}}(\cosh\sigma)$$

Proof :

It is known * that

$$K_p^{-m}(\cosh\sigma)=\frac{2^{1-m}.\ \sinh^{-m}\sigma}{\Pi(-\tfrac{1}{2})\Pi(m-\tfrac{1}{2})}.\int_0^{\sigma}\frac{\cos pu\,du}{(2\cosh\sigma-2\cosh\sigma)^{\frac{1}{2}-m}}$$

---

* Hobson's *Theory of Spherical and Ellipsoidal Harmonics*, p. 270, in formula 140 put $n=-\tfrac{1}{2}+pi$.

Hence

$$K_p^{-\frac{1}{2}}(\cosh\sigma)=\frac{\sqrt{2}}{\sqrt{\sinh\sigma}\,.\,\sqrt{\pi}}\,.\int_0^{\sigma}\cos pu\,du.$$

Therefore the series has the sum

$$\sqrt{\frac{2}{\pi}}\cdot\frac{1}{\sqrt{\sinh\sigma}}\int_0^{\sigma}\left\{\sum_{p=1}^{\infty}\frac{(-1)^{p-1}\cos(2p-1)u}{2p-1}\right\}du$$

$$=\sqrt{\frac{2}{\pi}}\,\frac{1}{\sqrt{\sinh\sigma}}\,\frac{\pi}{4}\int_0^{\sigma}du,\ \text{taking } 0<\sigma<\frac{\pi}{2}$$

$$=\sqrt{\frac{2}{\pi}}\cdot\frac{\pi}{4\sqrt{\sinh\sigma}}\cdot=\frac{1}{2}\sqrt{\frac{\pi}{2}}\cdot\frac{\sigma}{\sqrt{\sinh\sigma}}$$

(4) Taking $T_n^m(x)=e^{\frac{1}{2}m\pi i}\,.\,P_n^m(x)$

$$\sum_{n=0}^{m}(2n+1)\,.\,T_n^{-\frac{1}{2}}(\cos\theta)=\frac{4}{\sqrt{2\pi}\sin\theta}\cdot\frac{\sin^2(m+1)\frac{\theta}{2}}{\sin\frac{\theta}{2}},\ 0\leq\theta<\pi$$

Proof:

It is known * that

$$T_n^{-\frac{1}{2}}(\cos\theta)=\frac{4\sin(n+\frac{1}{2})\theta}{(2n+1)\sqrt{2\pi}\sin\theta},\ 0\leq\theta<\pi$$

Hence

$$\sum_{n=0}^{m}(2n+1)T_n^{-\frac{1}{2}}(\cos\theta)$$

$$=\frac{4}{\sqrt{2\pi}\,(\sin\theta)}\left[\sin\frac{\theta}{2}+\sin\frac{3\theta}{2}+\sin\frac{5\theta}{2}+\ldots+\sin(m+\tfrac{1}{2})\theta\right].$$

$$=\frac{4}{\sqrt{2\pi}\sin\theta}\,\frac{\sin^2\frac{(m+1)\theta}{2}}{\sin\frac{\theta}{2}}.$$

---

* MacRobert, *Phil. Mag.*, Series 7, Vol. XIV, Oct., 1932, p. 636. "Mehler—Dirichlet Integral, etc."

(5) As it is known that

$$T_n^{-\frac{1}{2}}(\cos\theta)=\frac{4\sin(n+\frac{1}{2})\theta}{(2n+1)\sqrt{2\pi}\sin\theta}$$

various sine series (Fourier) can be used to obtain infinite series of $T_n^{-\frac{1}{2}}(\cos\theta)$.

(6) To evaluate $\int_{-1}^{1} P_n^{\mu}(z).\ P_m^{\mu}(z).\ (1-z)^{m+n}dz$

It is known * that

$$\int_0^{\frac{\pi}{2}} F(-n,\ \mu+\beta+n\ ;\ \beta;\ \sin^2\phi).\ F(-m,\ \mu+b+m\ ;\ b\ ;\ \sin^2\phi)\times$$

$$\times\cos^{2\mu+1}\phi.\ \sin^{2\mu+2\beta+2b+2m+2n-3}\phi.\ d\phi$$

$$=(-1)^{m+n}\frac{\Gamma(\mu+m+n+1).\ \Gamma(\mu+\beta+b+m+n-1)(\mu+b+m)_n(\mu+\beta+n)_m}{2\Gamma(2\mu+b+\beta+2m+2n).\ (\beta)_n(b)_m}$$

Also

$$P_n^m(z)=\frac{1}{\Pi(-m)}\cdot\left(\frac{z+1}{z-1}\right)^{\frac{m}{2}}F\left(-n,\ n+1,\ 1-m,\ \frac{1-z}{2}\right)$$

In the above integral we put

$\beta=b=1-\mu$ and $\frac{1-z}{2}=\sin^2\phi$ and the integral becomes

$$\frac{1}{4}\int_{-1}^{1}[\Pi(-\mu)]^2\left(\frac{z-1}{z+1}\right)^{\mu}P_n^{\mu}(z).\ P_m^{\mu}(z).\ \left(\frac{1+z}{2}\right)^{\mu}\left(\frac{1-z}{2}\right)^{-\mu+m+n}dz$$

$$=(-1)^{m+n}\frac{\Gamma(\mu+m+n+1)\Gamma(1-\mu+m+n)(1+m)_n\times(1+n)_m}{2\Gamma 2(m+n+1)\ (1-\mu)_n(1-\mu)_m}$$

---

* W. N. Bailey : " Some definite integrals allied to an integral of Jacobi," the *Proc. L. M S.*, Vol. 30, p. 415, result No. (2. 8).

or

$$\int_{-1}^{1} P_n^{\mu}(z).\, P_m^{\mu}(z).\, (1-z)^{m+n} dz$$

$$=\frac{2^{m+n+1}\,(-1)^{m+n-\mu}}{\{\Pi(-\mu)\}^2}\cdot\frac{\Gamma(\mu+m+n+1)\,\Gamma(m+n+1-\mu)\,(1+m)_n(1+n)_m}{\Gamma 2(m+n+1)\quad(1-\mu)_n(1-\mu)_m}$$

(7) To evaluate $\int_{-1}^{1} P_m^{v}\,.\, P_n^{v}\,.\,(1+z)^{m+n} dz$

v being *a*+*ve* integer and m and n being unrestricted.

It is known * that

$$\int_{-1}^{1}(1+z)^{2v+m+n}(1-z)^{v}.\;C_n^{v+\frac{1}{2}}.\;C^{v+\frac{1}{2}}.\,dz$$

$$=\frac{2^{4v+m+n+1}.\,\Gamma(v+m+n+1).\;\Gamma(3v+m+n+1).}{\Gamma(2v+m+1).\;\Gamma(2v+n+1).\;\Gamma(v+m+1)}\times$$

$$\times\frac{\{\Gamma(v+1)\}^2\,\{\Gamma(2v+m+n+1)\}^2}{\Gamma(v+n+1)\;\Gamma 2(2v+m+n+1)}$$

Also v being an integer

$$\dagger\, C_{n-v}^{v+\frac{1}{2}}\;(z)=\frac{(z^2-1)^{-\frac{1}{2}v}}{(2v-1)(2v-3)..\,3.1}\,P_n^{v}\;(z)$$

So, in the above integral, putting for m and n, $m-v$ and $n-v$ respectively we have

$$\int_{-1}^{1}\frac{(1+z)^{v+m+n}.\;(1-z)^{v}.\,(z^2-1)^{-v}}{[(2v-1)(2v-3)...3,1]^2}.\;P_m^{v}.\,P_n^{v}.dz$$

$$=\frac{2^{m+n+1}\,\Gamma(m+n-v+1)\;\Gamma(m+n+v+1)\;\{\Gamma(v+1)\}^2\,\{\Gamma(m+n+1)\}^2}{\Gamma(m+v+1)\;\Gamma(v+n+1)\,\Gamma(m+1)\;\Gamma(n+1)\;\Gamma(2m+2n+2)}$$

---

* Bailey, *l.c.*, result 6. 2.

† Whittaker and Watson, *Modern Analysis*, 3rd edition, p. 329.

Hence

$$\int_{-1}^{1} \mathrm{P}_m^v . \mathrm{P}_n^v . (1+z)^{m+n} . dz$$

$$= \frac{[(2v-1)(2v-3)\ldots 3.1]^2 . 2^{m+n+1} \ \{\Gamma(v+1)\}^2 \{\Gamma(m+n+1)\}^2 .}{\Gamma(m+v+1)\ \Gamma(v+n+1)\ \Gamma(m+1)}$$

$$\frac{\Gamma(m+n-v+1)\ \Gamma(m+n+v+1)}{\Gamma(n+1)\Gamma(2m+2n+2)}$$

(8) $$\int_{-1}^{1} \mathrm{Q}_n^2 . dz = \frac{1}{2n+1} \left[\frac{\pi^2}{2} - \{\psi^2(n+1)\}(1+\cos^2 n\pi) \right]$$

where $$\psi^2(z) = \frac{d^2}{dz^2} \log \Gamma(z)$$

This was given by Nicholson * for integral values of n only.

Proof:

It is known that

$$\int_{-1}^{1} \mathrm{Q}_m . \mathrm{Q}_n . dz = \frac{[\{\psi(n+1)-\psi(m+1)\}(1+\cos m\pi . \cos n\pi) - \frac{\pi}{2}\sin(n-m)\pi]}{(m-n)(n+m+1)}$$

(see Ganesh Prasad "On Non-Orthogonal Systems of Legendre's Functions," p. 39, Vol. XII, the *Proc. Benares Math. Society*).

In this integral put $m = n$. We have

$$\int_{-1}^{1} \mathrm{Q}_n^2 dz = \frac{1}{2n+1} \left[\frac{\pi^2}{2} . \underset{n=m}{\mathrm{Lt}} \frac{\sin(m-n)\pi}{(m-n)\pi} \right.$$

$$\left. -(1+\cos^2 n\pi) \underset{n=m}{\mathrm{Lt}} \frac{\psi(m+1)-\psi(n+1)}{m-n} \right]$$

$$= \frac{1}{2n+1} \left[\frac{\pi^2}{2} - (1+\cos^2 n\pi) . \psi^2(n+1) \right]$$

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* Nicholson : *Phil. Mag.*, 8 series, Vol. XLIII, pp. 1-29, " Zonal Harmonics."

Note.—W. N. Bailey in his papers in the *Proc. of the Cambridge Phil. Soc.*, Vol. XXVII, parts 2 and 3, has given some interesting series and integrals involving associated Legendre functions. The following few results, which can be easily obtained, were suggested by his papers.

(*i*)
$$\frac{\tan^m\theta}{(\cos\theta)^{n+1}}=\sum_{r=0}^{\infty}\frac{2^m.(m+2r)(m+r-1)!\,\Gamma(m+2r+n+1)}{r!\,\Gamma(n+m+1)}\times$$
$$\times P_n^{-m-2r}(\cos\theta)$$

(*ii*)
$$\int_0^s\frac{(\sin t)^{m-1}}{(\cos t)^{m+n+1}}P_n\{\cos(s-t)\}.\,dt=$$
$$=\sum_{r=0}^{\infty}\frac{(m+r-1)!\,\Gamma(m+2r+n+1)}{r!\,\Gamma(n+m+1)}P_n^{-m-2r}(\cos s)$$

(*iii*)
$$\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\left(\frac{\sin\theta}{2}\right)^m.\;P_{m+k}^{m-v}(\cos\theta)=0$$

(*iv*)
$$\sum_{m=0}^{\infty}(-1)^m(v+2m).\,\Gamma(k+2m+1)P_{k-v}^{-v-2m}(\cos\theta).\,C_{2m}^{v}(\cos\phi)$$
$$=\frac{\Gamma(k+1)\,.\,\cos(k+1)\psi.\left(\frac{\sin\theta}{2}\right)^{v}}{\Gamma v\,.\,\{\cos^2\theta+\sin^2\theta.\,\cos^2\phi\}^{\frac{k+1}{2}}}$$

(*v*)
$$\frac{\Gamma(k+1).\,\sin(k+1)\psi.\left(\frac{\sin\theta}{2}\right)^{v}}{\Gamma(v)\,\{\cos^2\theta+\sin^2\theta.\,\cos^2\phi\}^{\frac{k+1}{2}}}=\sum_{m=0}^{\infty}(-1)^m(v+2m+1)$$
$$\Gamma(k+2m+2)$$
$$C_{2m+1}^{v}(\cos\phi)\times P_{k-v}^{-v-2m-1}(\cos\theta)$$

where $\tan\psi=\tan\theta.\,\cos\phi$

Bull. Cal. Math. Soc., Vol. XXV No. 1 (1933).

E. W. HOBSON
1856–1933

3

HOBSON*

BY

GANESH PRASAD

(Calcutta University)

Ernest William Hobson, whose death we are assembled to mourn to-day, was one of the most unassuming men whom I have known. In a letter † sent to me almost exactly a year ago, he wrote the following account of his life: "I was born in Derby in 1856, and received my earlier education at Derby School. After working at experimental Physics for several months in the Physical Laboratory of the Science and Art Department of South Kensington, I was elected to an open scholarship at Christ's College, Cambridge, where I graduated in 1878, as senior Wrangler, and have remained in Cambridge ever since. I retired from my Chair on 30th September last."

Before giving a fairly detailed analysis of the work of Hobson as a teacher, author and investigator, I proceed to fill up some gaps in, and add some details to, the account of his life as given above.

Soon after graduation, Hobson became a Fellow of his College and held that position up to the end of his life, having become a senior Fellow in 1885. With the Fellowship almost immediately came his position as an Assistant Tutor in Mathematics and he also worked for many years as a "coach," taking a limited number of private students. The years 1879-1910, during which he was a model of "plain living and high-thinking," saw him becoming Sc. D. in 1892, F. R. S. in June 1893, President of the London Mathematical Society for the period November 1900 to November 1902, Stokes Lecturer in 1903, President of the Mathematical and Physical Section of the British Association in 1910 and Sadleirian Professor of Pure Mathematics in the same year. In 1907 the Royal Medal of the Royal Society was awarded to him and in 1920 the De Morgan Medal of the London Mathematical Society. The Calcutta Mathematical Society made him an Honorary Member almost at its foundation in 1908 and and a similar honour was conferred on him by the Benares Mathematical Society in 1918. At the time of his death, Hobson held honorary doctorates from the Universities of Oxford, Dublin, St. Andrews, Aberdeen, Sheffield and Manchester, the foreign membership of the Leopoldinisch-Carolinische Academie der Naturforscher of Halle, the honorary membership of the Royal Irish Academy and the corresponding membership of the Accademia delle Scienze of Bologna.

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* Address delivered before the special meeting of the Calcutta Mathematical Society held on the 16th July, 1933.

† A facsimile of the second half of the letter will be found in Appendix (A) at the end.

Hobson's first book, *A Treatise on Plane Tingonometry*, which will be considered in detail later, first came out in 1891, in the second edition in 1897, in the third edition (revised and enlarged) in 1911 and that has been followed by four more editions. The committee of certain famous German Societies and Academies which planned the *Encyclopädie der Mathematischen Wissenschaften* commissioned him to write the mathematical portion, *viz.*, the first part, of the article on the conduction of heat. This part of the article was written by Hobson himself in German (and not translated from an English version into German) and came out in 1904. At the international Congress of mathematicians held at Cambridge in 1912, Hobson was a leading figure as a Secretary and was instrumental with Professor A. E. H. Love, the other Secretary, in promptly bringing out the *Proceedings* of that Congress. The first edition of Hobson's second book, *The Theory of the Functions of a Real Variable and the Theory of Fourier's Series* came out in one volume in 1907, to be followed later by a second edition in 2 volumes (Volume 1 in 1921 and Volume 2 in 1926) and by a third edition of Volume 1 in 1927. His third great book, *The Theory of Spherical and Ellipsoidal Harmonics* came out in 1931. Hobson also brought out two smaller books, *viz.*, *Squaring the Circle* (1913) and *The Domain of Natural Science* (1923), the latter constituting the series of Gifford Lectures delivered by him at Aberdeen. Two still smaller books of his are the lecture on Napier, *John Napier and the Invention of Logarithms, 1614* (1914), and *The Ideal Aim of Physical Science* (1925).

Some of the activities of Hobson, not referred to above, may be mentioned here. He joined the London Mathematical Society in November, 1887, and was on its council from 1892 to 1910 and again from 1912 to 1919. He was a Vice-President in 1896, 1897, 1902, 1903 and from 1917 to 1919. His devotion to that Society was great and he attended its meetings with great regularity: most of his papers are published in its *Proceedings*. He served on the Council of the Royal Society during the years 1903-1905 and 1928-1930. His part in the reforms of the Cambridge University was not by any means a mean one : he belonged to the band, including Forsyth, Baker and W. H. Young, which succeeded on the 2nd of February, 1907, in abolishing the order of merit in the Mathematical Tripos. The fight was a hard one and the final victory was achieved only by a small majority of 780 against 640. He belonged to the Liberal School in Politics and used to take some interest in political controversies before he became Professor.

The mathematical researches of Hobson, amounting to nearly 50 papers, published during the period of over 50 years stretching right up to a year or two of his death, may be conveniently reviewed by considering the papers in three parts. The *first* part will cover the period from 1879 to 1900 during which he was chiefly engaged in the study of what are called harmonic functions and their applications to Mathematical Physics. The *second* part will cover the period during which he worked chiefly on the theory of functions of a real variable and which probably began about 1900 and ended with 1924. During the *third* period, 1929-1931, he returned to his old love, *viz.*, harmonic functions.

The papers of the first period are the following which I will take up for consideration one after the other :—

I. "Proof of Rodrigues's theorem" (*Messenger of Mathematics*, 1879, s. 2, Vol. 9, pp. 53-54).

II. "On Fourier's theorem" (*Messenger of Mathematics*, 1881, s. 2, Vol. 11, pp. 11-14).

III. "On a radiation problem" (*Proceedings of the Cambridge Phil. Society*, 1888, Vol. 6, pp. 184-187).

IV. "Synthetic solutions in the conduction of heat" (*Proceedings of the London Mathematical Society*, 1888, s. 1, Vol. 19, pp. 279-294).

V. "On a class of Spherical Harmonics of complex degree with applications to physical problems (*Trans. of the Cambridge Phil. Society*, 1888, Vol. 14, pp. 211-236).

VI. "Systems of Spherical Harmonics" (*Proc. L. M. S.*, 1891, s. 1, Vol. 22, pp. 431-449).

VII. "The harmonic functions of the elliptic cone" (*Proc. L. M. S.*, 1892, s. 1, Vol. 23, pp. 231-240).

VIII. "On a theorem in differentiation and its application to Spherical Harmonics" (*Proc. L. M. S.*, 1892, s. 1, Vol. 24, pp. 55-67).

IX. "On the evaluation of a certain surface integral, and its application to the expansion in series of the potentials of ellipsoids" (*Proc. L. M. S.*, 1892, s. 1, Vol. 24, pp. 80-96).

X. "On Bessel's functions and relations connecting them with Hyperspherical and Spherical Harmonics" (*Proc. L. M. S.*, 1893, s. 1, Vol. 25, pp. 49-75).

XI. "On a theorem in the differential calculus" (*Messenger of Mathematics*, 1894, s. 2, Vol. 33, pp. 115-119).

XII. "On the most general solution of given degree of Laplace's equation" (*Proc. L. M. S.*, 1895, s. 1, Vol. 26, pp 492-494).

XIII. "On some general formulae for the potentials of ellipsoids, shells and discs" (*Proc. L. M. S.*, 1896, s. 1, Vol. 27, pp. 519-544).

XIV. "On a type of Spherical Harmonics of unrestricted degree, order and argument" (*Philosophical Transactions of the Royal Society of London*, 1896, Vol. 187A, pp. 443-531).

XV. "Note on some properties of Bessel's functions" (*Proc. L. M. S.*, 1897, s. 1, Vol. 28, pp. 370-375).

XVI. "On Green's function for a circular disc, with applications to electrostatic problems" (*Transactions of the Cambridge Philosophical Society*, 1900, Vol. 18, pp. 277-291).

(I)

This paper of 14 lines is reproduced here in its entirety on account of its being the first research publication of Hobson :—

"*Proof of Rodrigues's Theorem.*

Consider the expression

$$(x-1+h)^n(x+1+h)^n ;$$

the coefficients of h^{n-m} and h^{n+m} are respectively

$$\frac{1}{(n-m)\,!}\left(\frac{d}{dx}\right)^{n-m}(x^2-1)^n \text{ and } \frac{1}{(n+m)\,!}\left(\frac{d}{dx}\right)^{n+m}(x^2-1)^n.$$

Now

$$(x-1+h)^n=\sum_{r=0}^{r=n} a_r(x-1)^{n-r}h^r,$$

$$(x+1+h)^n=\sum_{r=0}^{r=n} a_r(x+1)^{n-r}h^r,$$

where $a_r=a_{n-r}$.

The coefficient of h^{n-m} in the product

$$=\sum_{t=0}^{t=n-m} a_{n-m-t}\;a_t(x-1)^{m+t}(x+1)^{n-t}$$

$$=(x-1)^m(x+1)^m\sum_{t=0}^{t=n-m} a_{m+t}\;a_{n-t}\;(x-1)^t(x+1)^{n-m-t}$$

$$=(x^2-1)^m \text{ coefficient of } h^{n+m},$$

therefore

$$\frac{1}{(n-m)\,!}\frac{d^{n-m}}{dx^{n-m}}(x^2-1)^n=(x^2-1)^m\frac{1}{(n+m)\,!}\frac{d^{n+m}}{dx^{n+m}}(x^2-1)^n."$$

The paper seems to have attracted much attention in Cambridge circles at the time: it is followed by a paper, "On Rodrigues's Theorem" (pp. 55-60), by Glaisher who gives a historical account of the theorem from 1816, when it was published by Rodrigues, right up to December, 1877, and mentions as many as 8 proofs. The 9th proof is Hobson's about which Glaisher says it "is very elegant and simple; and as far as I know it is new."

(II)

The paper begins with the following introductory paragraph:—

"I use here the notation employed in Todhunter's *Integral Calculus* in the reproduction of Poisson's investigation of the series which represents a function between assigned limits. It is often felt as a difficulty that there is a certain arbitrariness in first ma definitely small and afterwards making α and β so. Tl geometrical form of proof, which is, so far as I know, nev elucidate the necessity for considering the limit of g ε small compared with the limits of α and β."

Hobson remarks (on p. 18) that "a similar method sphere instead of a circle furnishes a proof of the exp function in a series of Laplace's coefficients."

It is not surprising that even a man of Hobson's c 1881 into Poisson's error of taking $\operatorname*{Lim}_{h=1} \sum_{0}^{\infty} a_n h^n$ to equal some relevant extracts, see Appendix (B) at the end.

(III) and (IV)

The second paper is introduced with the words: "T the present communication is to give the solutions ex definite integrals of certain problems in the variable mo in two and three dimensions, in which the boundaries of the body are straight edges or planes. The solutions are ol method which has been applied by Sir W. Thomson (s *Works*, Vol. 2) to some cases of conduction; this metho superimposing the temperature due to suitable distributions taneous or continuous sources. The effect of an initial of heat in such a body is obtained by regarding such initial as consisting of an infinite number of instantaneous po the heat from which is left to diffuse throughout the body. result is the solution of the problem of finding the of the infinite rectangle bounded by $x=a$, $x=-a$, the boundaries are maintained at zero temperature, and temperature over the plane is given by $\phi(x, y)$.

The solution is given by Hobson in the form

$$\frac{1}{8a\sqrt{\pi t}}\int_{-a}^{a}\int_{0}^{\infty}\left[e^{-(y-y')^2/4kt} - e^{-(y+y')^2/4kt}\right]$$

$$\times\left[\theta_3\left\{\frac{\pi(x-a)}{4a}, e^{-(\pi^2 kt)/4a^2}\right\}\right.$$

$$-\theta_3\left\{\frac{\pi(x+x'+2a)}{4a},\ e^{-(\pi^2 kt)/4a^2}\right\}\Big]\phi(x', y')\,dx'\,dy'.$$

The case of radiation from an infinite plane bounded by the axis of x across that axis is also considered. Analogues of the problems considered in two dimensions are given for the case of three dimensions.

The first paper deals with radiation and the results given in it are practically incorporated in the second paper.

(V)

" This paper presents an investigation of the properties of harmonics of degree, $-\frac{1}{2}-p\sqrt{-1}$, and their application to the discussion of potential problems when the bounding surfaces are cones and spheres. In particular the free electrification of two spherical conductors connected by a short wire of finite thickness is discussed."

(VI)

This important paper begins as follows: " The following communication has been suggested by a study of the theories of spherical harmonics given by Thomson and Tait and by Maxwell. I have, first of all, considered the most general solution of Laplace's equation, which is of degree 0, and have then shown how from this or the corresponding harmonics of degree -1, harmonics of any integral degree may be derived by differentiation with respect to a number of axes. I have further shown that all harmonics may be derived from harmonics of degree -1 by successive differentiation with respect to one axis only; new expressions for the ordinary tesseral harmonics have been obtained from this theorem.

" A theory of poles for the zonal and tesseral harmonics of the second kind is given, analogous to Maxwell's theory for those of the first kind. I have considered classes of harmonics which involve the circular measure of the azimuthal angle ϕ, and which I have ventured to call 'circulatory harmonics.' In the last section, I have obtained definite integral expressions for the most general harmonics of given order, irrespective of degree and have also given series involving an arbitrary function, which express the most general harmonics of the first kind of given order."

The headings of the various articles are as follow :—

The most general harmonics of given positive or negative degree ; Table of Harmonics of degree zero ; The system of Point-Harmonics ; The system of Line Harmonics ; Formulae for the Tesseral Harmonics of the first and second kind ; Circulatory Harmonics ; Cylindrical Harmonics ; The most general Spherical Harmonics of given order.

The point-harmonics, of which the simplest is

$$\frac{P_n(\cos\theta)}{r^{n+1}} = \frac{(-1)^n}{2^n n!}\frac{\partial^n}{\partial z^n}\frac{1}{r},$$

are derived from $\frac{1}{r}$; and in a similar manner line harmonics, of which the simplest is

$$\frac{Q_n(\cos\theta)}{r^{n+1}} = \frac{(-1)^n}{2^n n!}\frac{\partial^n}{\partial z^n}\left\{\frac{1}{r}\log\frac{r+z}{\sqrt{x^2+y^2}}\right\},$$

are derived from $\frac{1}{r}\log\sqrt{\frac{r+z}{r-z}}$.

The most general spherical harmonics of order m are

$$\begin{matrix}\cos\\ \sin\end{matrix}\, m\phi\,(x^2+y^2)^{\frac{m}{2}}\int_0^{\pi}\sin^{2m}\psi\, f(z+c\sqrt{x^2+y^2}\cos\psi)\,d\psi$$

$$+\begin{matrix}\cos\\ \sin\end{matrix}\, m\phi\,(x^2+y^2)^{\frac{m}{2}}\int_0^{\infty}\sinh^{2m}u\,\phi(z+c\sqrt{x^2+y^2}\cosh u)\,du$$

(VII)

Hobson says : " The harmonic functions for the circular cone were introduced by Mehler ; an account of his theory is given in Heine's *Kugelfunctionen*. In the present communication, I give some indications of a theory of the more general harmonic functions which are required for the corresponding potential problems connected with the elliptic cone ; I propose to call these harmonics *elliptic conal harmonics.*"

(VIII)

The differentiation theorem is the following: Let $f_n(x, y, z)$ be a homogeneous function of degree n, then

$$f_n\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\frac{1}{r} = \frac{(-1)^n(2n)!}{2^n n!}\frac{1}{r^{2n+1}}\left(1-\frac{r^2\nabla^2}{2.\,2n-1}\right.$$

$$\left.+\frac{r^4\nabla^4}{2.\,4.\,2n-1.\,2n-3}-\cdots\right)f_n(x, y, z).$$

Among the interesting applications are the proofs of the following theorems first given by Sir W. D. Niven :—

$$(2n)!\,P_n\left(\frac{xx_0+yy_0+zz_0}{rr_0}\right)$$

$$=2^n r^{n+1} r_0^{n+1}\left(\frac{\partial}{\partial x}\ \frac{\partial}{\partial x_0}+\frac{\partial}{\partial y}\ \frac{\partial}{\partial y_0}+\frac{\partial}{\partial z}\ \frac{\partial}{\partial z_0}\right)\frac{1}{rr_0}\ ;$$

$$G_n=\left(1-\frac{D^2}{2.\,2n-1}+\frac{D^4}{2.\,4.\,2n-1.\,2n-3}-\ldots\right)H_n,$$

where —

$$D^2=a^2\frac{\partial^2}{\partial x^2}+b^2\frac{\partial^2}{\partial y^2}+c^2\frac{\partial^2}{\partial z^2}\ ,$$

H_n = the solid spherical harmonic

$$\left\{\begin{matrix} & x & yz & \\ 1 & y & zx & xyz \\ & z & xy & \end{matrix}\right\}\Pi\left(\frac{x^2}{a^2+\theta}+\frac{y^2}{b^2+\theta}+\frac{z^2}{c^2+\theta}\right),$$

G_n = the corresponding ellipsoidal harmonic

$$\left\{\begin{matrix} & x & yz & \\ 1 & y & zx & xyz \\ & z & xy & \end{matrix}\right\}\Pi\left(\frac{x^2}{a^2+\theta}+\frac{y^2}{b^2+\theta}+\frac{z^2}{c^2+\theta}-1\right).$$

(IX)

Hobson says: " The formulae given by most writers on the subject of the attraction of ellipsoids, express the potentials in the form of definite integrals; such formulae have been given by Dr. Ferrers (*Q. J. M.*, Vol. 14) and recently in a very elegant form by Mr. Dyson (*Q. J. M.*, Vol. 25). The formulae given in the present communication are of such a character that approximate values of the potential may be obtained by taking as many terms of the series as may be necessary; whereas the definite integral formulae do not lend themselves readily to such approximation."

The surface integral theorem proved by Hobson is the following:

"

$$\iint Y_n(x, y, z) f(x, y, z) d\,S$$

$$=4\pi R^{2n+2} \frac{2^n n\,!}{(2n+1)\,!} \times$$

$$\left\{1+\frac{R^2\nabla^2}{2.\,2n+3}+\frac{R^4\nabla^4}{2.\,4.\,2n+3.\,2n+5}+\ldots\right\} Y_n\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) f(x, y, z),$$

where, on the right-hand side, x, y, z are all put equal to zero after the operations have been performed, and ∇^2 denotes the operator $\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$. The only restriction to which the function $f(x, y, z)$ is subject, is that it must be finite and continuous throughout the sphere." The integral is taken over the whole surface of the sphere

$$x^2+y^2+z^2=R^2.$$

Among the numerous applications of the theorem, may be mentioned the derivation of Sir W. D. Niven's expression for the external ellipsoidal harmonic in a series of spherical harmonics, and expansions of the potentials of ellipsoidal shells, solid ellipsoids, and elliptic discs of variable density, the law of force being any given function of the distance.

(X)

Hobson says about $J_m(r)$ and $J_{m+\frac{1}{2}}(r)$, m being an integer:—

" There is, however, another mode in which both kinds of functions may be considered to arise ; it appears, that if we consider the equation in p variables corresponding to

$$\nabla^2 V + V = 0,$$

the function $\dfrac{J_{\frac{1}{2}p-1}(r)}{r^{\frac{1}{2}p-1}}$ plays the same part in relation to this equation that $J_0(r)$ does in relation to the equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + V = 0,$$

and that thus $\dfrac{J_m(r)}{r^m}$ may be considered to be the Bessel's function of zero order when there are $2m+2$ variables, and also that $\dfrac{J_{m+\frac{1}{2}}(r)}{r^{m+\frac{1}{2}}}$ will be the Bessel's function of zero order when $2m+3$ is the number of the variables. In the present paper, various properties of the functions are developed from this point of view, the method having the advantage of dealing with both classes of functions at once. A considerable number of relations connecting the functions of different orders, both amongst themselves and with corresponding hyperspherical harmonics, are obtained, many of which are believed to be new. Many of these theorems arise from a comparison of different ways of expressing the same solution of one of the equations

$$\nabla^2 V = 0, \ \nabla^2 V + V = 0,$$

the number of variables being unrestricted. Expressions are obtained for the zonal and tesseral harmonics as definite integrals involving Bessel's functions."

Among the results the following may be mentioned here:—

$$P_n^m(\mu) = \frac{1}{(n-m)!}\int_0^\infty \lambda^n e^{-\lambda \cos\theta} J_m(\lambda \sin\theta)\, d\lambda ;$$

$$Q_n^m(\mu)=\frac{1}{(n-m)\,!}\int_0^\infty \lambda^n\, e^{-\lambda\cos\theta}\, Y_m(\lambda\sin\theta)\, d\lambda,$$

n and m being positive integers (including zero).

(XI)

The paper gives a general formula for

$f_n\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots \frac{\partial}{\partial x_p}\right) \mathrm{F}\left\{\phi(x_1, x_2, \dots x_p\right\}$, where f_n is an integral rational homogeneous function of the nth degree in the differential operators, the result being

$$\sum_{r=0}^{n} \frac{1}{(n-r)\,!}\;\frac{d^{n-r}\mathrm{F}}{d\phi^{n-r}}\, f_n\left(\frac{\partial}{\partial h_1}, \dots \frac{\partial}{\partial h_p}\right)\mathrm{P}^{n-r}$$

where F and ϕ are any functions,

$\mathrm{P}=\phi\,(x_1+h_1, x_2+h_2, \dots x_p+h_p)-\phi(x_1, x_2, \dots x_p)$ and $h_1, h_2, \dots h_p$ are all put equal to zero after the differentiation.

If $\phi=x_1^2+x_2^2+\dots+x_p^2\equiv\rho^2$, we have the result given in (VIII), *viz.*

$$f_n\left(\frac{\partial}{\partial x_1}, \dots \frac{\partial}{\partial x_p}\right) \mathrm{F}(\rho^2)$$

$$=\left\{2^n\frac{d^n\mathrm{F}}{d(\rho^2)^n}+\frac{2^{n-2}}{1\,!}\;\frac{d^{n-1}\mathrm{F}}{d(\rho^2)^{n-1}}\,\nabla^2+\dots\right\}f_n(x_1, \dots x_p)$$

where $\nabla^2=\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\dots+\frac{\partial^2}{\partial x_p^2}$.

(XII)

The result of the paper is the following :—

The most general solution of Laplace's equation of given (unrestricted) degree n is

$$f_n(x, y, z, r)=\left[1-\frac{r^2\nabla^2}{2.\,2n-1}+\frac{r^4\nabla^4}{2.\,4.\,2n+3.\,2n+5}-\dots\right]\phi_n(x, y, z)$$

$$+r^{2n+1}\left[1+\frac{r^2\nabla^2}{2.\,2n+3}+\frac{r^4\nabla^4}{2.\,4.\,2n+3.\,2n+5}+\dots\right]\frac{\psi_n(x,\,y,\,z)}{r^{n+1}};$$

ϕ_n, ψ_n being arbitrary functions of degree n, subject only to the condition that the above series are convergent.

(XIII)

The introduction contains the following: "The problem of the determination of the potential of solid ellipsoids and of ellipsoidal shells at an external or an internal point, is one which has engaged the attention of many celebrated mathematicians. Among the various methods by which the potential of a uniform solid ellipsoid of gravitating matter at an external or internal point has been found, perhaps the most ingenious and interesting is that of Dirichlet. His method is that of multiplying the integral to be found by a discontinuous factor, which factor is equal to unity throughout the ellipsoid and to zero at all external points; the volume integral taken throughout the ellipsoid, which expresses the potential, is thus replaced by an integral to be taken throughout all space, and the simplification thus made in the limits of integration facilitates the reduction of the volume integral to a single integral.........It occurred to me some three years ago that, by the use of a discontinuous factor of a more general form than that of Dirichlet, formulae might be obtained for the potentials of ellipsoids of variable density and with any law of force; and, in fact, the principal results given in the present communication were obtained by the use of the expression

$$\frac{i\,e^{-\frac{1}{2}(i r \pi)}}{2\sin r\pi}\cdot\int_{+\infty}^{\infty}(-\psi)^{r-1}e^{if\psi}\sin\,\psi\,d\psi,$$

in which the integral is taken along a path from ∞ on the positive side of the ψ-axis round the point $\psi=0$, to $\psi=\infty$ on the negative side of the ψ-axis; the real part of the above expression is equal to

$$\frac{\pi}{2\Gamma(1-r)}\cdot\frac{1}{(1-f)^r}\quad\text{or } 0$$

according as $f < 1$ or $f > 1$.

After reading the modified form of Dirichlet's method given by Kronecker (*Vorlesungen*, Vol. 1), in which the factor

$$\int_{-\infty}^{\infty} \frac{e^{c(q+iz)}}{q+iz}\, dz,$$

q denoting a positive quantity, is used, I found that my results could be obtained more simply by use of the expression

$$\int_{-\infty}^{\infty} \frac{e^{c(q+iz)}}{(q+iz)^{\lambda}}\, dz,$$

which is equal to

$$\frac{2\pi}{\Pi(\lambda-1)}\, c^{\lambda-1},$$

or zero, according as c is positive or negative, λ, q being positive quantities; this factor is accordingly used in the present paper."

The various headings in the paper are: The Potential of an n-dimensional Elliptic Disc which is in an $n+1$-dimensional space; the Law of Force being that of the inverse $(m+1)$th Power of the Distance; The Potential of an n-dimensional Solid Ellipsoid; The Potentials of infinitely thin Homoeoidal Rings and Ellipsoidal Shells; Extension of the Formulae for the Potentials of an Elliptic Disc and a solid Ellipsoid; The Potentials of Rings and Shells in case $m>n-2$; The Internal Potentials of Shells of Uniform Density; Extension of Formulae for the Potential of a Solid Ellipsoid.

(XIV)

[*] "The first attempt to define $P_n(z)$ by an expression which will be valid for unrestricted values of n and z was made, although unsuccessfully, by Heine.[†] Schläfli [‡] was the first to define $P_n(z)$ and $Q_n(z)$ for unrestricted n and z and the work achieved by him has

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[*] See Prasad's *Treatise on Spherical Harmonics and the Functions of Bessel and Lamé*, Part 2, p. 26.

[†] See his *Handbuch der Kugelfunktionen*, 1878, Bd. 1, pp. 37-38.

[‡] "Ueber die zwei Heinesche Kugelfunktionen mit beliebigem Parameter und ihre ausnahmslose Darstellung durch bestimmte Integrale" (Berne, 1881).

been described by him in the following words: 'I have succeeded in defining the two Heinian spherical harmonics for every arbitrary parameter by means of integrals which are valid without any exception and in deriving from those integrals, merely by the addition of the paths of integration, the chief relations between the different spherical harmonics. …… In the treatment of those definite integrals which represent solutions of differential equations, the intolerable defect has been met by me that one allows such solutions only under restrictive conditions for the constant elements of the problem; because one dreads to use curvilinear paths of integration. …… The omission of the free use of the path of integration has produced the effect, that one distinguishes, according to the discoverer, Laplace, Jacobi, Dirichlet, for the same function, several integral forms which, in respect of the situation of the path with reference to the critical points, are essentially one and the same and differ from one another only in the unessential form of the path.' "

Hobson admitted the accuracy of Schläfli's work regarding $P_n(z)$ and $Q_n(z)$ and extended it by the use of integrals with double circuits, first introduced in Analysis by Jordan and Pochhammer. This epoch-making achievement of Hobson has not been as yet materially improved upon. His own words deserve to be quoted here: "In the standard treatise of Heine, the forms and properties of the functions $P_n^m(\mu)$, $Q_n^m(\mu)$ are investigated for complex values of μ, the degree n and the order m being primarily real and integral; various extensions are made to cases in which n is not so restricted, but in default of a general definition of the functions for unrestricted values of n and m, these extensions are frequently, incomplete, and in some cases erroneous."

(XV)

This paper seems to have been hastily conceived, contrary to the usual practice of Hobson. The integral theorem, *viz.*

$$(n^2-m^2)\int_0^a \frac{J_m(x)J_n(x)}{x}\,dx$$

$$=(n-m)J_m(a)J_n(a)+a\{J_n(a)J_{m+1}(a)-J_m(a)J_{n+1}(a)\},$$

first given by Sonine (*Math. Ann.*, Vol. 16), is used to obtain some results about the roots of Bessel's functions, the most important of which, *viz.*, that the positive roots of the equations

$$J_n(x)=0 \text{ and } J_{n+1}(x)=0$$

occur alternately, had been proved earlier by E. B. Van Vleck (*A. M. J.*, Vol. XIX).

(XVI)

The paper begins as follows: "The main object of the present communication is to obtain the Green's function for the circular disc, and for the spherical bowl. The function for these cases does not appear to have been given before in an explicit form, although expressions for the electric density on a conducting disc or bowl under the action of an influencing point have been obtained by Lord Kelvin by means of a series of inversions. The method employed is the powerful one devised by Sommerfeld and explained fully by him in the paper referred to below. The application of this method given in the present paper may serve as an example of the simplicity which the consideration of multiple spaces introduces into the treatment of some potential problems which have hitherto only been attacked by indirect and more ponderous methods." The system of co-ordinates used is that first introduced by Neumann in 1864 for the treatment of the problem of the electric distribution in an anchor-ring; they are called peri-polar co-ordinates.

The papers of the second period are enumerated below; excepting XIX, all of them being from the *Proceedings of the London Mathematical Society*:—

(XVII)

"Non-uninform convergence, and the integration of series" (*s.* 1, Vol. 34, 1902, pp. 245-259).

(XVIII)

"On the infinite and the infinitesimal in mathematical Analysis" (*s.* 1, Vol. 35, 1903, pp. 117-140).

(XIX)

"On the integration of series" (*Acta Math.*, Vol. 27, 1903, pp. 209-216).

(XX)

"On modes of convergence of an infinite series of functions of a real variable" (*s.* 2, Vol. 1, 1904, pp. 373-387).

(XXI)

"On inner limiting sets of points in a linear interval" (*s.* 2, Vol 2, 1905, pp. 316-326).

(XXII)

"On the failure of convergence of Fourier's series" (*s.* 2, Vol. 3, 1905, pp. 48-61).

(XXIII)

"On the general theory of transfinite numbers and order types" (*s.* 2, Vol. 3, 1905, pp. 170-188).

(XXIV)

"On the arithmetic continuum" (*s.* 2, Vol 4, 1907, pp. 21-28).

(XXV)

"On absolutely convergent improper double integrals" (*s.* 2, Vol. 4, 1907, pp. 136-159).

(XXVI)

"On partial differential co-efficients and on repeated limits in general" (*s.* 2, Vol. 5. 1907, pp. 225-236).

(XXVII)

"On the uniform convergence of Fourier's series" (*s.* 2, Vol. 5, 1907, pp. 275-289).

(XXVIII)

"On repeated integrals" (*s.* 2, Vol. 5, 1907, pp. 325-334).

(XXIX)

"On a general convergence theorem, and the theory of the representation of a function by series of normal functions" (*s.* 2, Vol. 6, 1908, pp. 349-395).

(XXX)

"On the second mean-value theorem of the integral calculus" (*s.* 2, Vol. 7, 1909, pp. 14-23).

(XXXI)

"On the representation of a function by a series of Legendre's functions" (*s.* 2, Vol. 7, 1909, pp. 24-39).

(XXXII)

"Sir William Rowan Hamilton's fluctuating functions" (*s.* 2, Vol. 7, 1909, pp. 338-358).

(XXXIII)

"On the representation of a function by a series of Bessel's functions" (*s.* 2, Vol. 7, 1909, pp. 359-388).

(XXXIV)

"On change of the variable in a Lebesgue integral" (*s.* 2, Vol. 8, 1910, pp. 10-21).

(XXXV)

"On some fundamental properties of Lebesgue integrals in a two-dimensional domain" (*s.* 2, Vol. 8, 1910, pp. 22-39).

(XXXVI)

"On the fundamental lemma of the calculus of variations and on some related theorems" (*s.* 2, Vol. 11, 1913, pp. 17-28).

(XXXVII)

"On the representation of a summable function by a series of finite polynomials" (*s.* 2, Vol. 12, 1913, pp. 156-173).

(XXXVIII)

"On the convergence of series of orthogonal functions" (*s.* 2, Vol. 12, 1913, pp. 297-308).

(XXXIX)

"On the linear integral equation" (*s.* 2, Vol. 13, 1914, pp. 307-340).

(XL)

"On the representation of the symmetrical nucleus of a linear integral equation" (*s.* 2, Vol. 14, 1915, pp. 5-30).

(XLI)

"Theorems relating to functions defined implicitly, with applications to the calculus of variations" (*s.* 2, Vol. 14, 1915, pp. 147-177).

(XLII)

"On certain theorems in the theory of series of normal orthogonal functions" (*s.* 2, Vol. 14, 1915, pp. 428-439).

(XLIII)

"On Hellinger's integrals" (*s.* 2, Vol. 18, 1920, pp. 249-265).

(XLIV)

"On the summability of generalized Fourier's series" (*s.* 2, Vol. 22, 1924, pp. 420-424).

The third period of Hobson's activity as a researcher gave only the following papers, the first three of which were all published in series 2 of the *Proceedings of the L. M. S.* and the last in its *Journal*:—

(XLV)

"The addition theorem of general Legendre functions" (Vol. 29, 1929, pp. 355-381).

(XLVI)

"Generalization of two theorems of Stieltjes concerning Legendre's co-efficients" (Vol. 30, 1930, pp. 239-247).

(XLVII)

"On a generalization of Watson's expressions for Legendre functions" (Vol. 30, 1930, pp. 373-378).

(XLVIII)

"On the integration of trigonometrical series" (Vol. 2, 1927, pp. 164-166).

From the point of view of the lay public, two events stand out as special in the life of Hobson, *viz.*, his admission to the Royal Society of London in 1893 and his election to the Sadleirian Professorship in 1910. It will be therefore appropriate to consider here the position occupied by Hobson in the eyes of mathematicians in these two years.

I. *(a)* First of all as a researcher, Hobson had in 1893 ten papers to his credit, including the first two, which never found subsequent to their publication any mention in mathematical literature. The third and fourth papers were inspired by the works of Lord Kelvin and were practically nothing better than improvements on his results. Similar remark applies to the sixth paper which owed its origin to the treatment of spherical harmonics by Lord Kelvin and Maxwell. The fifth paper was inspired by the results of Neumann and Mehler about conal harmonics. The seventh paper had the same origin but did not lead to anything important. The eighth paper is based on a result of Clebsch. The ninth paper is based on the 8th and gives some interesting series for potentials of ellipsoids, the results including the known expressions given by Sir Frank Dyson and Sir W. D. Niven. The tenth paper is of importance for the novelty of the method as well as for the interesting expressions obtained for P_n^m and Q_n^m. Of these papers only two have been quoted in the *Encyklopädie der mathematischen Wissenschaften*; and although this record of Hobson compares favourably with the records of some of his contemporaries, who, although his juniors by some years, were admitted into the Royal Society several years before him, there is nothing in that record strikingly original like the theory of invariants of Cayley or the solution of the general quintic given by Hermite.

(b) Let us now consider Hobson's position as a teacher and author. When he joined the Cambridge University after passing the competitive examination for a scholarship at Christ's College he brought with him a sound knowledge of Statics, Dynamics, Algebra, Analytical Conics and Trigonometry of a standard only slightly lower than that of the present Calcutta B. A. (Honours) in those subjects. This type of freshman was by no means a rarity, as Maxwell had sought admission into the Cambridge University after graduating as M. A. from the Edinburgh University. But there is no doubt that Hobson was from his earliest years at Cambridge a marked man and the expectations which were formed of his success in the Mathematical Tripos

were fulfilled by his coming out as Senior Wrangler. In those days and up to comparatively recent times, a Senior Wrangler was almost a demi-god to his College authorities, and Hobson became a Fellow almost immediately after his Tripos Examination. He settled down in his College as an Assistant Tutor, giving lectures on the easier parts of the Tripos Examination and, as a "coach," taking a limited number of private students for the Tripos competetion, the income from fees for coaching being about £30 a year from each student. Hobson's success as a coach was considerable as some of his pupils became high Wranglers, but he never allowed his research work to suffer for love of income from his fees as a coach and he deliberately kept, unlike Routh, the number of his private students low. But his work as College lecturer and coach brought him face to face with difficulties in Mathematics, as felt by the student, and resulted in his bringing out his *Treatise on Plane Trigonometry*. In the Preface, he says: "In the present treatise, I have given an account, from the modern point of view, of the theory of circular functions, and also of such applications of those functions as have been usually included in works on Plane Trigonometry. It is hoped that the work will assist in informing and training students of Mathematics who are intending to proceed considerably further in the study of Analysis, and that, in view of the fulness with which the more elementary parts of the subject have been treated, the book will also be found useful by those whose range of reading is to be more limited .... The second part of the book, which may be supposed to commence at Chapter XIII, contains an exposition of the first principles of the theory of complex quantities; hitherto, the very elements of this theory have not been easily accessible to the English student, except recently in Prof. Chrystal's excellent treatise on Algebra. The subject of Analytical Trigonometry has been too frequently presented to the student in the state in which it was left by Euler, before the researches of Cauchy, Abel, Gauss and others, had placed the use of imaginary quantities and especially the theory of infinite series and products, where real or complex quantities are involved, on a fair scientific basis. In the chapter on the exponential theorem and logarithms, I have ventured to introduce the term 'generalized logarithm' for the doubly infinite series of values of the logarithm of a quantity."

The book produced along with Chrystal's *Algebra* a great impression on the English-knowing students—an impression like that produced by what may be called the parent book of Cauchy, *viz.*, *Algébrique*

Analyse of 1823, on the student public in France. Hobson's *Trigonometry* has had a wide circulation and there is no English book on that subject which is its equal in rigour.

The book, although in some respects similar to Chrystal's Algebra, the second volume of which came out in 1889, supplied a real need and has made the name of Hobson known far and wide in the student world. It may be safely concluded that by 1893, Hobson had established his reputation as a teacher and author.

II. *(a)* During the period of half a generation ending with his election to the Sadleirian Professorship, Hobson was at his best, thus belying the opinion of some observers, *e g.*, Ostwald, that the best work of a scientist or mathematician is accomplished before he is 35. In this period, Hobson published about 25 papers, at least three of which are epochmaking and secured for him the reputation of being in 1910 the greatest exponent of Pure Mathematics in the British Isles. These papers are (1) the memoir of 1896, "On a type of spherical harmonics of unrestricted degree, order and argument," (2) "On the uniform convergence of Fourier's series," 1907, and (3) "On the representation of a function by a series of Legendre's functions," 1909. On account of these researches, Hobson's name became widely known as an eminent mathematician throughout the world.

(b) But the greatest achievement of Hobson was the bringing out in 1907 of his treatise, *The Theory of Functions of a Real Variable and the Theory of Fourier's Series.* In the Preface, Hobson gives in the following words an insight into the origin of the book: "The course of study, of which the present treatise is the outcome, followed an order very similar to the historical order in which the subject was developed. Commencing with the study of Fourier's series, in their applications to the problems of Mathematical Physics, and provided with a knowledge of the Differential and Integral Calculus, of the traditional kind in which notions of the nature of continuity and of limits, founded on an uncritical use of intuitions of space and time, are the stock in trade, I was led, by the difficulties, connected with those series, and through an attempt to understand the literature which deals with them, to a study of the theories of real number, due to Cantor and Dedekind, and to that of the theory of sets of points. A study of the foundations of the Integral Calculus, and of the general theory of functions of a real variable, formed the natural continuation of the course. The present work has been written with the object of presenting in connected form, and of thus rendering more easily

accessible than hitherto, the chief results which are to be found scattered through a very large number of memoirs, periodicals and treatises. I have endeavoured, as far as possible, to fill up gaps in the various theories which occur in different parts of the subject. The proofs of the theorems have in many cases been simplified, often in accordance with developments of the theory later in date than the original proofs; other theorems have been given in a form more general than that in which they were first discovered."

As the very first course of lectures ever delivered at Cambridge on the theory of functions of a real variable was the course delivered by Hobson in 1902, it may be concluded that Hobson became devoted to the study of that subject about 1900. The course referred to above was attended by Professors H. F. Baker and G. B. Mathews, and three senior students including myself. The course was an event at Cambridge and well it might have been, as at the time the state of knowledge of the subject, specially in the British Isles, was deplorable, as the following words of Professor W. H. Young * prove :—

"At the turn of the century *Mengenlehre* was a term of which the meaning was all but unknown .... With exceptions indeed, but very few exceptions, the work of Georg Cantor, in so far as it had individually attracted attention, was regarded as an attempt of chiefly philosophical interest, and this attitude of mind can scarcely surprise us, since many of the more prominent mathematical professors of the day were as unenlightened for practical purposes with regard to the fundamental concepts of Analysis as Williamson, the author of well-known works on Differential and Integral Calculus, then still commonly used in our ancient Universities and not yet superseded in those parts of the globe in which the English influence is paramount. Williamson adopted the standpoint that a change of order in the procedure to the limit could not affect the result, and its study was for him of the nature of a purely metaphysical question.

"Even in France, one of the most prominent members of the Institute, known for his work in cognate matters, refused, as I have it from the brilliant mathematician Rene Baire at first hand, to believe that a function $f(x, y)$ of two variables which is continuous with respect to each variable separately, could have any point of discontinuity; I need hardly say that $f(x, y)$ may be continuous with respect to every straight line in the plane and yet present discontinuities."

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* See his presidential address. "On the progress of mathematical analysis in the twentieth century" (*Proc. L. M. S.*, s. 2, Vol. 24, 1925, pp. 421-434, specially pp. 422-424).

A vast change has come in this matter over the English-knowing mathematicians during the last twenty-five years and the fact that the concepts of analysis are widely and correctly known by professional mathematicians among them is mainly due to the work of Hobson.

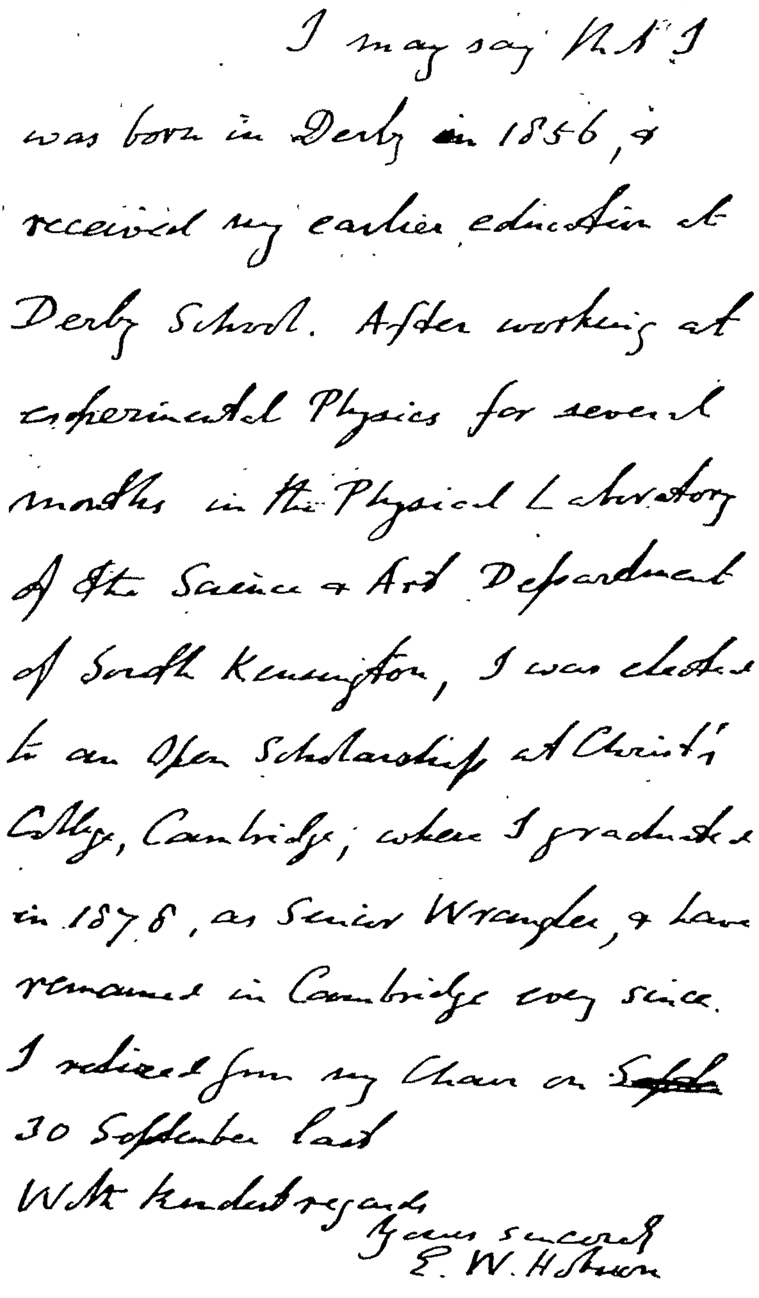
After 1910, for another 15 or 16 years, Hobson produced work of high order, chiefly in the field of the theory of functions of a real variable and its applications to other branches of Analysis. From 1929 he began to show the signs of advanced age. He died on the 18th April, 1933.

Hobson was married to a Swiss German lady in 1882 and left three surviving sons.

Before concluding, I may mention some peculiarities of Hobson. As a lecturer, Hobson was punctual in attendance, clear-headed in exposition and precise in language. In my time, 1899-1904, he would deliver his lecture standing during the whole hour; the lecture was not extempore and it was not exactly a dictation from notes; the notes were there either in his hand or on the tall desk before him, but they were consulted from time to time chiefly for lengthy formulae to be dictated. He was never averse to explaining difficulties on the black-board on which, not rarely, intricate workings were given. The lectures of Hobson were so instructive and full of substance that the auditor, who made full use of a course of them and reproduced in his note-book all that he had heard, secured thereby almost a complete treatise on the subject of the course. For examples, I may mention the lectures on "Theory of Vibrations and Sound" in 1899 and the lectures on "Spherical Harmonics and Bessel Functions" in 1900.

As Sadleirian Professor, Hobson generally delivered two courses of lectures every year, each course consisting of nearly 25 lectures. It is worth noting here that some of these courses were on the history of Mathematics. As examples, I may mention the course on the history of the problem of the "squaring of the circle" and that on the history of the invention of the calculus; these courses were delivered respectively in 1912 and 1914.

APPENDIX (A)



I may say that I was born in Derby ~~on~~ in 1856, & received my earlier education at Derby School. After working at experimental Physics for several months in the Physical Laboratory of the Science & Art Department of South Kensington, I was elected to an Open Scholarship at Christ's College, Cambridge; where I graduated in 1878, as Senior Wrangler, & have remained in Cambridge ever since. I retired from my Chair on ~~Septr~~ 30 September last

With kindest regards

Yours sincerely

E. W. Hobson

APPENDIX (B)

(I)

[Extract from Todhunter's *Integral Calculus*, 1921, pp. 298-299.]

308. The following expansion may be obtained by ordinary trigonometrical methods :

$$\frac{1-h^2}{1-2h\cos\frac{\pi(v-x)}{l}+h^2}=1+2h\cos\frac{\pi(v-x)}{l}+2h^2\cos\frac{2\pi(v-x)}{l}+\ \dots \quad (1)$$

h being less than unity, so that the series is convergent.

Multiply both sides of (1) by $\phi(v)$ and integrate with respect to v between the limits $-l$ and l; also make h approach to unity as its limit. On the right-hand side the different powers of h become ultimately unity. The numerator of the fraction on the left-hand side will ultimately vanish, and thus the integral would vanish *if the denominator of the fraction were never zero.* But if x lies between $-l$ and l, the term $\cos\frac{\pi(v-x)}{l}$ will become equal to unity during the integration, and thus the denominator of the fraction will be $(1-h)^2$ and will tend towards zero as h approaches unity. Hence the integral will not necessarily vanish ; we proceed to ascertain its value.

Let $v-x=z$ and $h=1-g$, then

$$\int\frac{(1-h^2)\,\phi\,(v)dv}{1-2h\cos\frac{\pi(v-x)}{l}+h^2}=\int\frac{g(1+h)\,\phi\,(x+z)dz}{g^2+4h\sin^2\frac{\pi z}{2l}}.$$

Now the only part of the integral which has any sensible value, is that which arises from very small positive or negative values of z; thus we may put

$$\sin\frac{\pi z}{2l}=\frac{\pi z}{2l}$$

and $$\phi(x+z) = \phi(x);$$

and the integral becomes

$$g(1+h)\,\phi(x)\int \frac{dz}{g^2+h\frac{\pi^2 z^2}{l^2}} = 2g\phi(x)\int \frac{dz}{g^2+\frac{\pi^2 z^2}{l^2}}$$

$$= \frac{2l\,\phi(x)}{\pi}\ \tan^{-1}\frac{\pi z}{gl}.$$

Suppose a and $-\beta$ to be the limits of z; we thus get

$$\frac{2l\,\phi(x)}{\pi}\left\{\tan^{-1}\frac{\pi a}{gl}+\tan^{-1}\frac{\pi\beta}{gl}\right\}.$$

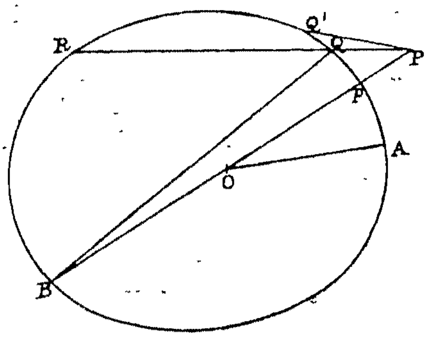
Hence, finally, when g is supposed to vanish, we have $2l\ \phi(x)$.

Therefore if x lies between $-l$ and l,

$$\phi(x) = \frac{1}{2l}\int_{-l}^{l}\phi(v)\,dv + \frac{1}{l}\sum_{1}^{\infty}\int_{-l}^{l}\phi(v)\cos\frac{n\pi(v-x)}{l}\,dv \qquad (2)$$

(II)

[Extract From Hobson's paper "on Fourier's Theorems."]



"Let P be a point at a distance unity from the centre O of a circle of radius h.

Let θ be the angle which any radius O Q makes with a fixed radius O A.

Let $f(\theta)$ be a function regarded as known at every point on the circumference between $\theta = 0$ and $\theta = \pi$.

Let P O A be θ'.

We shall seek the value of the integral

$$\int_0^{\pi}\frac{1-h^2}{1-2h\cos(\theta-\theta')+h^2}\,f(\theta)\,d\theta.$$

Suppose P Q meets the circumference of the circle again in R, then the expression under the integral sign is equal to

$$\frac{PQ.\,PR.}{PQ^2} f(\theta) = \frac{PR}{PQ} f(\theta).$$

Let Q Q′ be an element $h\,d\,\theta$ and let P O meet the circumference again in B.

Then $$< Q\,P\,Q' = \frac{Q\,Q' \cos O\,Q\,R}{P\,Q} = \frac{h d\theta}{PQ} \cdot \frac{PR - PQ}{2h}$$

$$= \tfrac{1}{2}\left(\frac{PR}{PQ}\, d\theta - d\theta \right);$$

hence $$\frac{PR}{PQ}\, d\theta = 2(d\alpha + d\mu)$$

if $$d\alpha = < Q\,B\,Q',\ d\mu = < Q\,P\,Q',$$

thus an element of the integral becomes

$$2f(\theta)\,(d\alpha + d\mu).$$

Now suppose h to become indefinitely near unity, that is, suppose P indefinitely near to the circle, then for all positions of Q at a finite distance from P we may consider P to be absolutely upon the circumference, in which case

$$\alpha + \mu = \frac{\pi}{2},$$

and therefore

$$d\alpha + d\mu = 0,$$

thus the corresponding element of the integral vanishes.

But if Q is not at a finite distance from P, since P is not absolutely upon the circumference of the circle, for such elements $d\alpha + d\mu$ is not zero. We must integrate for these separately and in order to include all such elements we must integrate along an arc which, though indefinitely small, is infinitely great compared with the distance of P from the circumference of the circle. We are then sure to have included all points in which $d\alpha + d\mu$ is not zero.

This arc subtends an indefinitely small angle at B but an angle π at P, as it is when looked at from P an infinite straight line. Hence, if $f(\theta)$ is continuous for the value $\theta=\theta'$, the value of the integral is $2\pi f(\theta')$.

... ...

Let $\theta=\frac{\pi v}{l}$, $\theta'=\frac{\pi x}{l}$, $f(\theta)=\phi(x)$.

Then since

$$\frac{1-h^2}{1-2h\cos(\theta-\theta)+h^2}=1+2\sum_1^\infty h^n\cos n(\theta-\theta'),$$

we have since

$$d\theta=\frac{\pi}{l}dv$$

the theorem that

$$\frac{1}{2l}\int_0^l\phi(v)dv+\frac{1}{l}\sum_1^\infty\int_0^l\phi(v)\cos\frac{n\pi(v-x)}{l}dv,$$

is equal to $\phi(x)$ for any value of x between 0 and l for which the function is continuous ".* ... ...

Bull. Cal. Math. Soc., Vol. XXV, No. 1 (1933).

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* For a careful and correct consideration of Poisson's method, see Hobson's *Theory of functions of a real variable and theory of Fourier's series*, Vol. 2, Art. 474.

EXTRACTS

FROM THE

CONSTITUTION OF THE SOCIETY

10 Every person, desirous of admission into the Society as a Member, must be proposed and recommended by at least two members.

11 Every recommendation of a proposed Member must be delivered to the Secretary and read at one of the ordinary meetings of the Society. * * * * * * * *

14 Every person elected as a Member, shall pay his admission fee within one month, and the first annual contribution within three months of the day of his election; otherwise his election shall be void; but the Council shall, in particular cases, have the power of extending the period within which the first annual contribution shall be paid.

15 * * The admission fee shall be ten rupees in all cases; the annual contribution shall be twelve rupees in the case of Members residing in Calcutta or its Suburbs and Howrah (who shall be called Resident Members), and six rupees in the case of Non-resident Members. * * * * *

16 Members who have paid the admission fee and a sum of one hundred rupees in advance shall become Life Members.

Authors of Papers intended for communication to the Society are requested to draw up Abstracts of their Papers, to be read at the meetings of the Society and to be circulated if possible, the Abstracts to indicate the nature of the methods employed and the character of the results obtained.

A Paper accepted for publication in the Bulletin of the Society should not be re-published by the author in a different Journal without the sanction of the Council and without making the necessary acknowledgment.

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MARCH, 1933

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